

## Solutions to tutorial exercises set(0) : (Analysis 1)

### Exercise 01:

A. All these numbers are real, so they are  $\mathbb{R}$ :

(i)  $-2$  is a negative natural number, i.e. an integer  $(a, b, c, d)$

(ii)  $\frac{1}{3}$  is a ratio of integers so it is a rational  $(c, d)$

(iii)  $0$  is an integer  $(a, c, d)$ .

(iv)  $7$  is a natural number and an integer. It is in fact also a prime number - That is, only divisible by itself or  $1$ . It is also an odd number (can not be exactly divided by  $2$ )

$(a, c, d, g)$

(v)  $\frac{21}{5}$  is a rational number (actually an improper fraction)  
 $(c, d)$ .

numerator  $(21)$ , denominator  $(5)$

(vi)  $-\frac{3}{4}$  is a rational number (a proper fraction)  $(b, c, d)$

numerator  $(-3)$ , denominator  $(4)$

(vii)  $0.73$  is a actually a decimal representation of a rational number ( $0.73 = \frac{73}{100}$ ) sometimes called a decimal fraction, or simply a decimal;  $(c, d, f)$ .

(viii)  $11$  is a natural number and an integer (like  $7$  it is also prime, and is also odd as any prime greater than  $2$  must be,

$(a, c, d, g)$

(ix)  $8$  is another natural number and an integer (but it is not prime) since it can be written as  $2 \times 2 \times 2 = 2^3$ . It is also an even number  
 $(a, c, d)$ .

(x) The square root of  $2$  is not a rational number.  $(d, e)$

(xi)  $-0.49$  is a decimal representation of the negative rational number  $(-\frac{49}{100})$   $(b, c, d, f)$

(xii)  $\pi$ , the ratio of the circumference of a circle to its diameter, is not a rational number (it is an irrational number). That is, it cannot be written as a fraction.  $\frac{22}{7}$  for example, is just an approximation to  $\pi$ ; (d, e).

B. (i)  $0 \times 1 = 0$ , i.e. zero - which is also finite

(ii)  $0 + 1 = 1$ , finite, non-zero

(iii)  $\frac{1}{0}$  does not exist - it is not infinite, negative, zero, finite or non-zero (b).

(iv)  $2 - 0 = 2$ , finite and non-zero (e, f).

(v)  $0^2 = 0 \times 0 = 0$ , zero and finite (d, e).

(vi)  $0 - 1 = -1$ , negative, finite, non-zero (c, e, f).

(vii)  $\frac{0}{0}$  does not exist (you can't "cancel" the zeros!). It is not infinite, negative, zero, finite or non-zero - it just does not exist (b).

(viii)  $\frac{3}{0} \times 0 + \frac{3}{0}$  does not exist because  $\frac{3}{0}$  does not exist (b).

(ix)  $\frac{0^3}{0}$  again, does not exist (b)

(x)  $\frac{2}{2} = 1$  - no problem here, finite and non-zero (e, f)

### Exercise 02%

1)  $x > 0$

2)  $1 < x < 0$

3)  $-1 < x < 3$

4)  $-2 \leq x < 2$

5) If the absolute value of  $x$  is less than 2 then this means that if  $x$  is positive then  $0 \leq x < 2$ , but if  $x$  is negative then we must have  $-2 < x \leq 0$ . So, combining these we must have  $-2 < x < 2$ . This can also be expressed in terms of modulus  $|x| < 2$ .

### Exercise 03:

$$\begin{aligned} \text{(i)} \quad \frac{2}{x+1} - \frac{3}{x-2} &= \frac{2(x-2) - 3(x+1)}{(x+1)(x-2)} \\ &= \frac{2x - 4 - 3x - 3}{x^2 - 2x + x - 2} \\ &= \frac{-x - 7}{x^2 - x - 2} \\ &= -\frac{x+7}{x^2 - x - 2} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \frac{1}{x-1} + \frac{1}{x+1} - \frac{1}{x+2} &= \frac{(x+1) + (x-1)}{(x-1)(x+1)} - \frac{1}{(x+2)} \\ &= \frac{x+1+x-1}{(x^2-1)} - \frac{1}{(x+2)} \\ &= \frac{2x(x+2) - (x^2-1)}{(x^2-1)(x+2)} \\ &= \frac{2x^2 + 4x - x^2 + 1}{x^3 + 2x^2 - x - 2} \\ &= \frac{x^2 + 4x + 1}{x^3 + 2x^2 - x - 2} \end{aligned}$$

### Exercise 04:

odd function:  $f(-x) = -f(x)$

even function:  $f(-x) = f(x)$

$$\text{(i)} \quad 3x^3 - x = f(x) \Rightarrow f(-x) = 3(-x)^3 - (-x) = -(3x^3 - x) = -f(x)$$

$3x^3 - x$  is an odd function

$$\text{(ii)} \quad \frac{x^2}{1+x^2} = f(x) \Rightarrow f(-x) = \frac{(-x)^2}{1+(-x)^2} = f(x)$$

$\frac{x^2}{1+x^2}$  is an even function

$$\text{(iii)} \quad \frac{2x}{x^2-1} = f(x) \Rightarrow f(-x) = \frac{2(-x)}{(-x)^2-1} = -\frac{2x}{x^2-1} = -f(x)$$

$$(iv) \frac{x^2}{x-1} = f(x) \Rightarrow f(-x) = \frac{(-x)^2}{(-x)-1} = \frac{x^2}{-x-1}$$

$\frac{x^2}{x-1}$  neither an even function nor an odd function.

### Exercise 05:

(a) Let us suppose  $\frac{a}{b}$  as a number (if it exists) such that  $b \cdot x = a$ , then  $\frac{0}{0} = x$ ; where  $0 \cdot x = 0$  but this is true for all numbers.  
 $\Rightarrow$  it is undefined.

(b) Similar to (a), if we define  $\frac{1}{0}$  as a number (if it exists) such that  $0 \cdot x = 1$ , we reach the conclusion that there is no satisfied this condition.

### Exercise 06:

An odd integer:  $n = 2k + 1$  where  $k \in \mathbb{Z}$

we have:  $(2k + 1)^2 = 4k^2 + 4k + 1$ . Notice that, we can write:

$$4k^2 + 4k = 2(2k^2 + 2k) = 2p \text{ where } p = 2k^2 + 2k$$

So:  $(2k + 1)^2 = 2p + 1$ , which is an odd number.

### Exercise 07:

We assume by contradiction that  $\sqrt{2} \in \mathbb{Q}$  then:

$$\exists p, q \in \mathbb{Z}, \text{ gcd}(p, q) = 1 \text{ such that } \sqrt{2} = \frac{p}{q} \Leftrightarrow p^2 = 2q^2$$

*greatest common divisor*

$\Rightarrow 2$  divides  $p$  since the square of an integer is even only if the integer itself is even

$$\text{so } p = 2k \text{ where } k \in \mathbb{Z} \Rightarrow 4k^2 = 2q^2 \Rightarrow 2k^2 = q^2 \Rightarrow 2 \text{ divides } q^2 \Rightarrow$$

$2$  divides  $q$ , which is a contradiction since  $p$  and  $q$  are relatively prime.

### Exercise 08:

since  $\frac{p}{q}$  is a root, on substituting in the given equation, we find:

$$a_0 \left(\frac{p}{q}\right)^n + a_1 \left(\frac{p}{q}\right)^{n-1} + a_2 \left(\frac{p}{q}\right)^{n-2} + \dots + a_{n-1} \left(\frac{p}{q}\right) + a_n = 0$$

multiplying by  $q^n$ :  $a_0 p^n + a_1 p^{n-1} q + a_2 p^{n-2} q^2 + \dots + a_{n-1} p q^{n-1} + a_n q^n = 0$  (\*)

dividing by  $p$ :  $a_0 p^{n-1} + a_1 p^{n-2} q + \dots + a_{n-1} q^{n-1} = -\frac{a_n q^n}{p}$

Notice that,  $a_0 p^{n-1} + a_1 p^{n-2} q + \dots + a_{n-1} q^{n-1}$  is an integer

$\Rightarrow -\frac{a_n q^n}{p}$  is also integer. Then since  $p$  and  $q$  are relatively prime  
 $\Rightarrow p$  does not divide  $q^n$  exactly and  
 $\Rightarrow$  so must divide  $a_n$

In the other hand, by transposing the first term of (\*):

$$a_1 p^{n-1} q + a_2 p^{n-2} q^2 + \dots + a_{n-1} p q^{n-1} + a_n q^n = -a_0 p^n$$

and dividing by  $q$ :

$$a_1 p^{n-1} + a_2 p^{n-2} q + \dots + a_{n-1} p q^{n-2} + a_n q^{n-1} = -\frac{a_0 p^n}{q}$$

We find that:  $a_1 p^{n-1} + a_2 p^{n-2} q + \dots + a_{n-1} p q^{n-2} + a_n q^{n-1} \in \mathbb{Z}$

so  $-\frac{a_0 p^n}{q} \in \mathbb{Z}$

$\Rightarrow q$  must divide  $a_0$  (since  $p$  and  $q$  are relatively prime so  
 $q$  does not divide  $p^n$  exactly and  
 so must divide  $a_0$ )

### Exercise 09:

Let us take:  $x = \sqrt{2} + \sqrt{3} \Rightarrow x^2 = (\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6}$

$\Leftrightarrow x^2 - 5 = 2\sqrt{6}$  and squaring  $(x^2 - 5)^2 = x^4 - 10x^2 + 25 = 4 \times 6 = 24$

$\Rightarrow x^4 - 10x^2 + 1 = 0$

We use the result in exercise 08. Notice that,  $a_0 = 1$  and  $a_n = 1$

so if  $\frac{p}{q}$  is a root of  $x^4 - 10x^2 + 1 = 0$ , we have:

$p$  must divide 1  $\Rightarrow p = \pm 1$

$q$  must divide 1  $\Rightarrow q = \pm 1$

Then, the only possible rational roots of the equation

$x^4 - 10x^2 + 1 = 0$  are  $\pm 1$  and these do not satisfy the equation

$\Rightarrow \sqrt{2} + \sqrt{3}$  cannot be rational number.

### Exercise 10:

Let us assume  $a < b$  adding  $a$  to both side:  $2a < a+b \Leftrightarrow a < \frac{a+b}{2}$

In the other hand, adding  $b$  to both sides  $a+b < 2b$  and  $\frac{a+b}{2} < b$

Thus  $a < \frac{a+b}{2} < b$ .

Let us now, prove that  $\frac{a+b}{2} \in \mathbb{Q}$ : Use take:  $\begin{cases} a = \frac{p}{q} \\ b = \frac{r}{s} \end{cases}$

where  $p, q, r, s \in \mathbb{Z}$

$$\Rightarrow \frac{a+b}{2} = \frac{1}{2} \left( \frac{p}{q} + \frac{r}{s} \right) = \frac{1}{2} \left( \frac{ps+qr}{qs} \right) = \frac{ps+qr}{2qs}$$

where  $\begin{cases} (ps+qr) \in \mathbb{Z} \\ \text{and} \\ 2qs \in \mathbb{Z} \end{cases}$

$$\Rightarrow \frac{ps+qr}{2qs} \in \mathbb{Q}$$