POWER SERIES

Power Series :

1) **Definition :**

We call power serie any serie of function $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$

we write : $\displaystyle\sum_{n=0}^{\infty} a_n x^n = \sum_{n\geq 0} U_n$ And $(a_n)_{n\geq 0}$ is a real sequence.

$U_n = a_n x^n$ is called the general term of the serie $\{S_n\}$.

Examples : $\displaystyle\sum_{n\geq 0} x^n$, $\displaystyle\sum_{n\geq 0} \frac{n}{n+1} x^n$, $\displaystyle\sum_{n\geq 0} 3^n x^n$

2) **Radius of convergence :**

The radius of convergence of a power serie $\displaystyle\sum_{n=0}^{\infty} a_n x^n$

Is the positive real R such that

$\begin{cases} \text{If } |x| < R \text{ the serie } \{S_n\} \text{ is convergent.} \\ \text{If } |x| > R \text{ the serie } \{S_n\} \text{ is divergent.} \end{cases}$

3) **Domain of convergence :** We call domain of convergence of a power serie $\{S_n\}$ the set of all reals where the serie is convergent.

If $|x| < R$, $\{S_n\}$ converges and the domain of convergence is the interval of the center zéro and the radius R $\quad D = \{ x \in \mathbb{R} \ / \ |x| < R\}$.

Theorem :

Let $\displaystyle\sum_{n\geq 0} a_n x^n$ a power serie. And R its radius of convergence. We have :

$\begin{cases} R = 0 \iff D = \{0\} \\ R = +\infty \iff D = \mathbb{R} \\ 0 < R < +\infty \iff D =]-R \quad R[\end{cases}$

Remark :

1) For $x = \pm R$ We don't conclude about convergence of the série $\{S_n\}$.

2) If the power serie is in the form $\displaystyle\sum_{n\geq 0} a_n (x - x_0)^n$ Then :

$$\begin{cases} R = 0 \iff D = \{x_0\} \\ R = +\infty \iff D = \mathbb{R} \\ 0 < R < +\infty \iff D =]-R + x_0 \quad R + x_0 [\end{cases}$$

4) Tests of convergence : (Calculation of radius R)

$$\text{Let } \sum_{n \geq 0} a_n x^n \text{ a power serie.}$$

a)   Test of Cauchy- Hadamard

If $\lim_{n \to +\infty} \sqrt[n]{|a_n x^n|} = \lim_{n \to +\infty} \sqrt[n]{|a_n|} \ |x| < 1 \ \left(\text{The serie } \sum_{n \geq 0} a_n x^n \text{ converges} \right).$

The radius of convergence R is given by : $R = \dfrac{1}{l}$ with $l = \lim_{n \to +\infty} \sqrt[n]{|a_n|}$

Examples :

1) $\displaystyle\sum_{n \geq 0} (2x)^n,$

2) $\displaystyle\sum_{n \geq 1} \left(\dfrac{x}{n}\right)^n$

b) Test of D'Alembert :

If $\lim_{n \to +\infty} \left| \dfrac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n \to +\infty} \left| \dfrac{a_{n+1}}{a_n} \right| \ |x| < 1 \ \left(\text{The serie } \sum_{n \geq 0} a_n x^n \text{ converges} \right).$

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Then the radius of convergence R given by : $R = \dfrac{1}{l}$. with $l = \lim\limits_{n \to +\infty} \left| \dfrac{a_{n+1}}{a_n} \right|$

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Examples : 1) $\displaystyle\sum_{n \geq 1} \dfrac{x^n}{n + 1}$

2) $\displaystyle\sum_{n \geq 0} n! \, x^n$

5) Sum of the power serie :

Consider $\displaystyle\sum_{n \geq 0} a_n x^n$ a power serie of the radius R.

We call $S_n = \displaystyle\sum_{k=0}^{n} a_k x^k$ The partial sum of the serie $\displaystyle\sum_{n \geq 0} a_n x^n$.

We call sum of the serie $\displaystyle\sum_{n \geq 0} a_n x^n$ a limit of its partial sum .

Written : $S = \lim\limits_{n \to +\infty} S_n$.

Example : $\displaystyle\sum_{n\geq 0}\left(\frac{x}{2}\right)^n$

6) Derivation and integration of power series :

Theorem :

Consider $\displaystyle\sum_{n\geq 0} a_n x^n$ a power serie , with radius R and the sum S. Then :

(1) The series $\displaystyle\sum_{n\geq 1} na_n x^{n-1}$ and $\displaystyle\sum_{n\geq 0}\frac{a_n}{n+1}x^{n+1}$, obtained by derivation.

And integration of the serie $\displaystyle\sum_{n\geq 0} a_n x^n$, have the same radius of convergence.

(2) $\displaystyle S'(x) = \left(\sum_{n\geq 0} a_n x^n\right)' = \sum_{n\geq 1} na_n x^{n-1}$

(3) $\displaystyle\int_0^x S(t)dt = \int_0^x \left(\sum_{n\geq 0} a_n t^n\right)dt = \sum_{n\geq 0}\frac{a_n}{n+1}x^{n+1}$

Example : $\displaystyle S = \sum_{n\geq 2}\frac{x^n}{(n-1)n}$

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7) Power series expansion (developement) :

Consider $f : D \subseteq \mathbb{R} \to \mathbb{R}$ Derivable on D. and $x_0 \in D$

We call the serie $\displaystyle\sum_{n=0}^{+\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ The Taylor serie of f in the neighborhood of x_0.

Theorem :

Any derivable function on the interval D is equal to the sum

of a power serie converges in this intervale.

Examples :   Developement (expansion) in the neighborhood of zero $(x_0 = 0)$.

1) $f(x) = e^x$

2) $f(x) = \sin(x)$

3) $f(x) = \cos(x)$

4) $f(x) = sh(x) =$, $g(x) = ch(x) =$

On a
$$\begin{cases} e^x = \sum_{n\geq 0} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots \\ e^{-x} = \sum_{n\geq 0} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^n}{n!}x^n + \cdots \end{cases}$$

$$sh(x) = x + \frac{1}{3!}x^3 + \cdots + \frac{1}{(2n+1)!}x^{2n+1} + \cdots = \sum_{n\geq 0} \frac{1}{(2n+1)!}x^{2n+1}$$

$$ch(x) = 1 + \frac{1}{2}x^2 + \cdots + \frac{1}{(2n)!}x^{2n} + \cdots = \sum_{n\geq 0} \frac{1}{(2n)!}x^{2n}$$

5) $f(x) = ln(1+x)$

We have : $\dfrac{1}{1+x} = 1 - x + x^2 + \cdots + (-1)^n x^n + \cdots = \sum_{n\geq 0}(-1)^n x^n$

$$f(x) = ln(1+x) = \int_0^x \frac{du}{1+u} = \sum_{n\geq 0} \frac{(-1)^n}{n+1}x^{n+1}$$

Examples :

1) $f(x) = \dfrac{x+2}{x^2 + 2x - 3}$

$ln(4 + x) = ?$