I. Numerical series

1) Definitions :

Consider the sequence $(u_n)_n$, $u_n \in IK$ $(IK = \mathbb{R}$ ou $\mathbb{C})$.
And (S_n) the sequence defined by : $S_n = u_0 + u_1 + u_2 + \cdots + u_n$
We call numérical serie the pair (S_n, u_n) formed by the sequences (u_n) and (S_n).
We call u_n the general term of the serie (S_n, u_n) .

$S_n = \sum_{k=0}^{k=n} u_k$ Called the partial sum of the serie (S_n, u_n) .

We say that serie (S_n, u_n) is convergent if and only if the sequence $(S_n)_n$ is convergent.

We call $s = \lim_{n\to+\infty} S_n = \sum_{n=0}^{+\infty} u_n$ the Sum of serie(S_n, u_n) .

A numérical serie that is not convergent, is said divergent. ($\lim_{n\to+\infty} S_n = \infty$ or does not exist.)

Examples :

1) $s = \sum_{n=0}^{+\infty} u_n$, $u_n = \dfrac{1}{(n+1)(n+2)}$

2) $\sum_{n=0}^{+\infty} ln\left(1+\frac{1}{n}\right)$

2) Necessary condition of convergence :

$$\sum_{n=0}^{+\infty} u_n \quad \text{converges} \quad \Rightarrow \quad \lim_{n\to+\infty} u_n = 0$$

Remark :

$$\lim_{n\to+\infty} u_n = 0 \quad \not\Rightarrow \quad \sum_{n=0}^{+\infty} u_n \quad \text{convergente}$$

Examples :

1) $\sum_{n=0}^{+\infty} \frac{n}{n+1}$

2) $\sum_{n=0}^{+\infty} ln\left(1+\frac{1}{n}\right)$

3) Particular series

1. Riemann' serie : $\sum_{n\geq 0} \frac{1}{n^{\alpha}}$ Converges iff $\alpha > 1$.
2. Harmonic serie : $\sum_{n\geq 0} \frac{1}{n}$ Diverges (Riemann α=1).
3. Geometric serie : $\sum_{n\geq 0} r^{\alpha}$ Converges iff $|r|<1$.
4. Serie of Bertrand : $\sum_{n\geq 0} \frac{1}{n^{\alpha}(\ln(n))^{\beta}}$ converges iff : $\begin{cases} \alpha > 1 \text{ et } \beta\in R \\ \beta > 1 \text{ et } \alpha=1 \end{cases}$

4) Convergence criteria (Tests of convergence) :

Let the serie : $\sum_{n\geq 0} u_n$ with a general term u_n.

Theorem 01 : Test of Cauchy

$$\text{If } \lim_{n\to+\infty} \sqrt[n]{|u_n|} = l \text{ Exist, Then : } \begin{cases} \sum_{n\geq 0} u_n \text{ converges if } l < 1 \\ \sum_{n\geq 0} u_n \text{ diverges if } l > 1 \end{cases}$$

Examples :

1) $\sum_{n\geq 0} \left(\frac{n+5}{2n+1}\right)^n$

2) $\sum_{n\geq 0} \left(\frac{2}{3}\right)^{n^2}$

Theorem 02 : Test of D'Alembert

$$\text{If } \lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right| = l \text{ Exist , Then } \begin{cases} \sum_{n\geq 0} u_n \text{ converges if } l < 1 \\ \sum_{n\geq 0} u_n \text{ diverges if } l > 1 \end{cases}$$

Examples :

1) $\sum_{n\geq 0} \frac{2^n}{n!}$

2) $\sum_{n\geq 0} \frac{(n+1)!}{n^2}$

Theorem 03 Test of comparison

$$\text{Let : } \sum_{n\geq 0} u_n \text{ and } \sum_{n\geq 0} v_n \text{ two series with positive terms .}$$

$$\text{For } n \geq n_0 , \quad u_n \leq v_n \text{ Then : } \begin{cases} \sum_{n\geq 0} v_n \text{ Converges} \Rightarrow \sum_{n\geq 0} u_n \text{ Converges} \\ \sum_{n\geq 0} u_n \text{ Diverges} \Rightarrow \sum_{n\geq 0} v_n \text{ Diverges} \end{cases}$$

Examples :

1) $\sum_{n\geq 0} \frac{3^n}{1+5^n}$

2) $\sum_{n\geq 2} \frac{1}{\sqrt{n^2-1}}$

Theorem 04 : Test of equivalence

Let : $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ two series with positive terms.

If $\lim_{n\to+\infty} \frac{u_n}{v_n} = 1$ Then $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ have the same behavior.

Examples :

1) $\sum_{n\geq 0} \frac{2n^2+1}{n^5+4}$

$\sum_{n\geq 0} \frac{7^n}{5^n+\ln(n)}$

Theorem 05 : Test of Abel

The serie $\sum_{n\geq 0}(u_n v_n)$ Converges if and only if $\begin{cases} (u_n) \text{ monotonic sequence and } \lim_{n\to+\infty} u_n = 0 \\ \exists M \geq 0 \quad \left|\sum_{k=0}^{n} v_k\right| \leq M \end{cases}$

Example :

$$\sum_{n\geq 0} \frac{(-1)^n}{2n^3+1}$$

5) Properties :

Let : $\sum_{n\geq 0} u_n$, $\sum_{n\geq 0} v_n$ numerical series, and α real number. We have :

$$\begin{cases} \text{If } \sum_{n\geq 0} u_n \text{ and } \sum_{n\geq 0} v_n \text{ convergs , Then } \sum_{n\geq 0} \alpha u_n \text{ and } \sum_{n\geq 0}(u_n+v_n) \text{ are convergents} \\ \text{If } \sum_{n\geq 0} u_n \text{ converges and } \sum_{n\geq 0} v_n \text{ diverges , then } \sum_{n\geq 0}(u_n+v_n) \text{ is divergent} \\ \text{If } \sum_{n\geq 0} u_n \text{ diverges and } \sum_{n\geq 0} v_n \text{ diverges, we don't conclure} \end{cases}$$

Example :

Let $\sum_{n\geq 0}\left(\frac{1}{5^n}+2^n\right)$ and $\sum_{n\geq 0}\left(\frac{1}{5^n}-2^n\right)$ diverges series.

6) Alternating serire :

We call alternating serie every serie given by the form :

$$\sum_{n\geq 0}(-1)^n u_n \quad \text{where } (u_n) \text{ have a constant sign.}$$

Theorem of Leibnitz :

If : $\begin{cases} \text{The sequence } (u_n) \text{ is monotonic} \\ \lim_{n\to+\infty} u_n = 0 \end{cases}$ Then $\sum_{n\geq 0}(-1)^n u_n$ is converges.

Examples :

1) $\sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n}}$

2) $\sum_{n\geq 1}(-1)^n sin\left(\frac{1}{n}\right)$

7) Absolute convergence and conditional convergente :

Let : $\sum_{n\geq 0} u_n$ a numerical serie.

a) The serie $\sum_{n\geq 0} u_n$ is absolutely convergente iff the serie $\sum_{n\geq 0}|u_n|$ is convergent.

b) The serie $\sum_{n\geq 0} u_n$ is conditionaly-converges iff the serie $\sum_{n\geq 0} u_n$ converges and the serie $\sum_{n\geq 0}|u_n|$ diverges.

Proposition :

$$\sum_{n\geq 0} |u_n| \text{ converges} \Longrightarrow \sum_{n\geq 0} u_n \text{ converges.}$$

Examples :

1) $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$

2) $\displaystyle\sum_{n\geq 0} \frac{(-1)^n}{2n^3 + n^2 + 3}$