Solution of Series of Exercises 4

Solution of Exercise 1

1. Let f be a periodic function defined by: $f(x) = x^2$, for $x \in [-\pi, \pi]$.

2. Since f is a periodic function an it is a piecewise C^1 on $\mathbb{R}$, then according to Dirichlet theorem, f is expandable into Fourier series.

2. f is even, so $b_n = 0, \forall n \geq 1$, and $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^\pi \dfrac{x^2}{2}dx = \dfrac{\pi^2}{3}$.

Integrating by parts twice, we get: $a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi \dfrac{x^2}{2}\cos(nx)dx = \dfrac{2}{n^2}(-1)^n$.

The Fourier series is then given by:

$$S_F(f) = \frac{\pi^2}{6} + 2\sum_{n\geq 1} \frac{(-1)^n}{n^2}\cos(nx).$$

3. $*x = \pi$ is a point of continuity, so

$$f(\pi) = S(\pi) \Leftrightarrow \frac{\pi^2}{2} = \frac{\pi^2}{6} + 2\sum_{n\geq 1}\frac{1}{n^2},$$

which implies that $\sum_{n\geq 1}\dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.

* $x = 0$ is also a point of continuity. So

$$f(0) = S(0) \Leftrightarrow 0 = \frac{\pi^2}{6} + 2\sum_{n\geq 1}\frac{(-1)^n}{n^2},$$

which implies that $\sum_{n\geq 1}\dfrac{(-1)^{n+1}}{n^2} = \dfrac{\pi^2}{12}$.

* Using Parseval's formula, we get:

$$\frac{2}{\pi}\int_0^\pi \frac{x^4}{4}dx = \frac{\pi^4}{10} = \frac{a_0^2}{2} + \sum_{n=1}^{+\infty} a_n^2.$$

i.e; $\sum\limits_{n\geq 1}\dfrac{1}{n^4} = \dfrac{\pi^4}{90}$.

Solution of Exercise 2

1. f is 2π-periodic and it is piecewise C^1 on $\mathbb{R}$, according to Dirichlet theorem f is expandable into Fourier series.

2. f is odd, so $a_n = 0, \forall n \geq 0$.
$b_n = \dfrac{2}{\pi} \int\limits_0^\pi x \sin(nx)dx$.
Integrating by parts, we get:

$$b_n = \frac{2}{\pi}(\left[-\frac{x}{n}\cos(nx)\right]_0^\pi + \frac{1}{n}\int\limits_0^\pi \cos(nx)dx) = \frac{2}{n}(-1)^{n+1}.$$

The Fourier series of f is then

$$S_f(x) = 2\sum_{n\geq 0} \frac{(-1)^{n+1}}{n}\sin(nx).$$

3. $x = \dfrac{\pi}{2}$ is a point of continuity, so $f(\dfrac{\pi}{2}) = S(\dfrac{\pi}{2})$.
i.e; $\dfrac{\pi}{2} = 2\sum\limits_{n\geq 0} \dfrac{(-1)^{2n+2}}{2n+1}(-1)^n$.
So $\sum\limits_{n\geq 0} \dfrac{(-1)^n}{2n+1} = \dfrac{\pi}{4}$.

4. Using Parseval's formula, we get: $\dfrac{2}{\pi}\int\limits_0^\pi x^2 dx = \dfrac{2\pi^2}{3} = \sum\limits_{n=1}^{+\infty} b_n^2$.

Hence $\sum\limits_{n\geq 1} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$.
On the other hand,

$$\begin{aligned} \sum_{n\geq 1}\frac{1}{n^2} &= \sum_{n\geq 1}\frac{1}{(2n)^2} + \sum_{n\geq 0}\frac{1}{(2n+1^2} \\ &= \frac{1}{4}\sum_{n\geq 1}\frac{1}{n^2} + \sum_{n\geq 0}\frac{1}{(2n+1^2}. \end{aligned}$$

So $\sum\limits_{n\geq 0} \dfrac{1}{(2n+1^2} = \sum\limits_{n\geq 1}\dfrac{1}{n^2} \underset{n\geq 1}{-} \dfrac{1}{4}\sum \dfrac{1}{n^2} = \dfrac{\pi^2}{8}$.