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Fourier Series

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Contents

Chapter 1

Fourier series

1.1 Periodic functions

Definition 1.1.1. *Let f be a function defined on $\mathbb{R}$. We say that f is periodic with period T (or $T-$periodic), if and only if:*

$$f(x+T) = f(x), \text{ for all } x \in \mathbb{R}. \tag{1.1}$$

Corollary 1.1.1. *Let f be a $T-$periodic function. We then have the following properties:*
1. The number $-T$ is also a period of f.
2. The number nT, $n \in \mathbb{Z}$ is also a period of f.

Proof. 1. Since f is a $T-$periodic function, we can then write:

$$\begin{aligned} f(x-T) &= f(x-T+T) \\ &= f(x), \text{ for } x \in \mathbb{R}. \end{aligned} \tag{1.2}$$

2. If $n \geq 0$, by recurrence, we can write:

$$\begin{aligned} f(x+nT) &= f(x+(n-1)T+T) = f(x+(n-1)T) \\ &= \ldots \\ &\ldots \\ &\ldots \\ &= f(x+T) = f(x), \text{ for all } x \in \mathbb{R}. \end{aligned} \tag{1.3}$$

Now, suppose that $n < 0$. Since $-T$ is also a period of f, using the previous proof gives:

$$f(x+nT) = f(x+(-n)(-T)) = f(x), \text{ for all } x \in \mathbb{R}. \tag{1.4}$$

□

Proposition 1.1.1. *Let f be a T-periodic function, then the function g defined on $\mathbb{R}$ by $g(x) = f(\alpha x + \beta)$ $(\alpha \neq 0)$ is $\frac{T}{\alpha}$- periodic.*

Proof. Since f is $T-$periodic, we can write

$$\begin{aligned} g\left(x+\frac{T}{\alpha}\right) &= f\left[\alpha\left(x+\frac{T}{\alpha}\right)+\beta\right] \\ &= f(\alpha x+\beta+T) = f(\alpha x+\beta) = g(x). \end{aligned} \tag{1.5}$$

□

Proposition 1.1.2. *Let f be a $T-$periodic function and integrable on an interval $[\lambda, \lambda+T]$ (interval of length T), then we have:*

$$\int_{\lambda}^{\lambda+T} f(x)dx = \int_0^T f(x)dx, \textit{ for all } \lambda \in \mathbb{R}. \tag{1.6}$$

Proof. Indeed:

$$\begin{aligned} \int_{\lambda}^{\lambda+T} f(x)dx &= \int_{\lambda}^{T} f(x)dx + \int_{T}^{\lambda+T} f(x)dx \\ &= \int_{\lambda}^{T} f(x)dx + \int_{T}^{\lambda+T} f(x-T)dx \ \text{ (because } -T \text{ is also a period of} f \text{)} \\ &= \int_{\lambda}^{T} f(x)dx + \int_{0}^{\lambda} f(x)dx \text{ (by posing } x-T=X\text{)} \\ &= \int_{0}^{T} f(x)dx, \text{ for all } \lambda \in \mathbb{R}. \end{aligned} \tag{1.7}$$

□

1.2 Trigonometric series

Definition 1.2.1. *A trigonometric series is a series of functions that can be written in the form:*

$$\frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right), \tag{1.8}$$

where $(a_n)_n$ and $(b_n)_n$ are two sequences of scalars and $T = 2l$ is the period of the series.

Definition 1.2.2. *(Complex form of a trigonometric series) Since $\sin 0 = 0$, we can assume that $b_0 = 0$.*

$$c_n = \frac{a_n - ib_n}{2} \text{ and } c_{-n} = \frac{a_n + ib_n}{2}, \tag{1.9}$$

the expression (1.8) can be rewritten in the complex form:

$$\sum_{n\in\mathbb{Z}} c_n \exp\left(i\frac{n\pi x}{l}\right). \tag{1.10}$$

Thus, a trigonometric series can be considered as a series of functions in $\mathbb{C}$ of the form $\sum_{n\in\mathbb{Z}} c_n \exp\left(i\frac{n\pi x}{l}\right)$.

1.2.1 Rules of convergence

Proposition 1.2.1. *1) If the series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} b_n$ are absolutely convergent, then the trigonometric series (1.8) is normally (therefore uniformly) convergent on $\mathbb{R}$, and its sum is a continuous function on $\mathbb{R}$.*
2) If the sequences (a_n) and (a_n) are positive, decreasing and converge to 0, the trigonometric series (1.8) simply converges on $\mathbb{R} - 2l\mathbb{Z}$ and it is uniformly convergent on any interval of the form $[2k\pi + \lambda, 2(k+1)\pi - \lambda]$, where $k \in \mathbb{Z}$ and $0 < \lambda < \pi$.

Proof. 1) We can write:

$$\left|a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right| \leq |a_n| + b_n, \text{ for all } x \in \mathbb{R}. \tag{1.11}$$

The trigonometric series (1.8) is therefore normally convergent on $\mathbb{R}$ and therefore uniformly convergent on $\mathbb{R}$ by the Weierstrass criterion.

The continuity of the sum of this series occurs because the functions $x \mapsto a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$ are continuous and the series (1.8) is uniformly convergent.

2) This is an immediate consequence of Abel's criterion. □

1.2.2 Calculation of coefficients of a trigonometric series

Proposition 1.2.2. *Suppose that the numerical series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} b_n$ are absolutely convergent, and let S be the sum of the trigonometric series (1.8). Then the coefficients of this series are given by:*

$$a_n = \frac{1}{l}\int_{-l}^{l} S(x)\cos\left(\frac{n\pi x}{l}\right)dx,\ n \geq 0, \tag{1.12}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} S(x)\sin\left(\frac{n\pi x}{l}\right)dx,\ n \geq 1, \tag{1.13}$$

Proof. We multiply the general term of the trigonometric series (1.8) by $\cos\left(\frac{m\pi x}{l}\right)$ or by $\sin\left(\frac{m\pi x}{l}\right)$ and under the following inequalities:

$$\left|\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]\cos\left(\frac{m\pi x}{l}\right)\right| \leq |a_n| + |b_n|,\ x \in \mathbb{R}, \tag{1.14}$$

$$\left|\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]\sin\left(\frac{m\pi x}{l}\right)\right| \leq |a_n| + |b_n|,\ x \in \mathbb{R}, \tag{1.15}$$

we can ensure the uniform convergence of the following series of functions:

$$\sum_{n\geq 1}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]\ \cos\left(\frac{m\pi x}{l}\right) \tag{1.16}$$

and

$$\sum_{n\geq 1}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]\sin\left(\frac{m\pi x}{l}\right) \tag{1.17}$$

Therefore, we can integrate term by term.

The properties (1.12) and (1.13) are therefore consequences of the following calculation:

$$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = 0,\ n \neq m, \tag{1.18}$$

$$\int_{-l}^{l} \cos^2\left(\frac{n\pi x}{l}\right)dx = \int_{-l}^{l} \sin^2\left(\frac{n\pi x}{l}\right)dx = l, \tag{1.19}$$

$$\int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = \int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx = 0,\ n, m \geq 1. \tag{1.20}$$

□

We can now define the Fourier series of a periodic function.

1.3 Fourier series

Definition 1.3.1. *Let f be a function defined on $\mathbb{R}$, $2l-$periodic and integrable on the interval $[-l, l]$. We call the Fourier series of f and we denote $SF(f)$ the trigonometric series:*

$$\frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right). \tag{1.21}$$

The numbers a_n and b_n defined by:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx,\ n \geq 0, \tag{1.22}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx,\ n \geq 1, \tag{1.23}$$

are called Fourier coefficients of f .

Corollary 1.3.1. *To calculate the Fourier coefficients of a function, we can calculate the integrals over any interval of the type $[\lambda, \lambda+2l]$ instead of $[-l, l]$, and the Fourier coefficients become:*

$$a_n = \frac{1}{l}\int_{\lambda}^{\lambda+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx,\ n \geq 0, \tag{1.24}$$

$$b_n = \frac{1}{l}\int_{\lambda}^{\lambda+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx,\ n \geq 1, \tag{1.25}$$

Proof. This is an immediate consequence of the Proposition 1.1.2 . □

1.3.1 Fourier series of even or odd functions

Proposition 1.3.1. *1) If f is even on $\mathbb{R}$, the Fourier coefficients are given by:*

$$\begin{cases} b_n = 0, \text{ for } n \geq 1 \\ a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \; n \geq 0. \end{cases} \tag{1.26}$$

2) If f is odd on $\mathbb{R}$, the Fourier coefficients are given by:

$$\begin{cases} a_n = 0, \; n \geq 0, \\ b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \text{ for } n \geq 1. \end{cases} \tag{1.27}$$

Proof. 1) Since f is even, we can then write:

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ &= \frac{1}{l}\left[\int_{-l}^{0} f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] \\ &= \frac{1}{l}\left[\int_{l}^{0} f(-x)\cos\left(-\frac{n\pi x}{l}\right)d(-x) + \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] \\ &= \frac{1}{l}\left[\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] \\ &= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx. \end{aligned}$$

On the other hand, if f is even on $\mathbb{R}$, the function $f\sin\left(\frac{n\pi}{l}x\right)$ is odd for all $n \geq 1$, and therefore:

$$\begin{aligned} b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = -\frac{1}{l}\int_{l}^{-l} f(x)\sin\left(\frac{n\pi x}{l}\right)d(-x) \\ &= -\frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)d(x) = -\frac{1}{l}b_n, \end{aligned}$$

Therefore $b_n = 0$, for all $n \geq 1$.

2) The proof is analogous for the case where f is odd. □

Example 1.3.1. *Consider the f 2π -periodic function defined by:*

$$f(x) = x, \; x \in \left]-\pi, \pi\right[. \tag{1.28}$$

f is odd, so for all $n \geq 0$, $a_n = 0$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = 2\frac{(-1)^{n-1}}{n},\ n \geq 1. \tag{1.29}$$

The Fourier series of f is therefore:

$$SF(f(x)) = 2\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sin(nx). \tag{1.30}$$

Example 1.3.2. *Consider the f 2π -periodic function defined by:*

$$f(x) = x^2,\ x \in [-\pi,\pi]. \tag{1.31}$$

f is even, so for all $n \geq 0$, $b_n = 0$, and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = 4\frac{\pi^2}{3},\ a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = 4\frac{(-1)^n}{n},\ n \geq 1. \tag{1.32}$$

The Fourier series of f is therefore:

$$SF(f(x)) = \frac{2}{3}\pi^2 + 4\sum_{n\geq 1}\frac{(-1)^n}{n^2}\cos(nx). \tag{1.33}$$

1.3.2 Riemann-Lebesgue Lemma (Necessary Convergence Condition)

Lemma 1.3.1. *Let f be a function integrable on an interval $[a,b]$, we then have:*

$$\lim_{n\to+\infty}\int_a^b f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \lim_{n\to+\infty}\int_a^b f(x)\sin\left(\frac{n\pi x}{l}\right)dx = 0.$$

Proof. It is enough to show that:

$$\lim_{n\to+\infty}\int_a^b f(x)\exp\left(i\frac{n\pi x}{l}\right)dx = 0. \tag{1.34}$$

Suppose that f is integrable on $[a,b]$, that is, that there exists a subdivision of the interval $[a,b]$ by a finite number of points:$a = x_0 < x_1 < < x_n = b$ and a staircase function φ defined by $\varphi(x) = m_j$, $x \in \left[x_{j-1}, x_j\right[,\ j = 1,2,...,n$, such that:

$$\left|f(x) - \varphi(x)\right| \leq \frac{\epsilon}{2(b-a)},\ \text{for all } \epsilon > 0. \tag{1.35}$$

We can write :

$$\int_a^b \left|[f(x)-\varphi(x)]\exp\left(i\frac{n\pi x}{l}\right)\right|dx \leq \int_a^b \left|f(x)-\varphi(x)\right| < \epsilon, \text{ for all } \epsilon > 0. \quad (1.36)$$

Therefore:

$$\begin{aligned}
\lim_{n\to+\infty}\int_a^b f(x)\exp\left(i\frac{n\pi x}{l}\right)dx &= \lim_{n\to+\infty}\int_a^b \varphi(x)\exp\left(i\frac{n\pi x}{l}\right)dx \\
&= \lim_{n\to+\infty}\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} m_j\exp\left(i\frac{n\pi x}{l}\right)dx \\
&= \sum_{j=1}^{n}\lim_{n\to+\infty}\int_a^b m_j\exp\left(i\frac{n\pi x}{l}\right)dx \\
&= \sum_{j=1}^{n}\lim_{n\to+\infty}\left[\frac{m_j l}{n\pi}\exp\left(i\frac{n\pi x}{l}\right)\right]_a^b \\
&= 0 \qquad (1.37)
\end{aligned}$$

□

Corollary 1.3.2. *Let f be a function defined on $\mathbb{R}$, $2l-$periodic and integrable on $[-l,l]$, then the sequences of Fourier coefficients (a_n) and (a_n) converge to 0 when $n \to +\infty$.*

1.3.3 Dirichlet Theorem (Sufficient Convergence Condition)

In this section, we will study a case of convergence of Fourier series.

Definition 1.3.2. *Let f be a function defined on an interval $[a,b]$. We say that f is piecewise continuous on $[a,b]$, if there exists a subdivision $\left\{\left[x_{j-1},x_j\right[,\ j=1,2,...,n\right\}$ of $[a,b]$ such that:*

1. f is continuous on each interval $\left[x_{j-1},x_j\right[$, $j=1,2,...,n$.

2. f admits discontinuities of the first kind at the points x_j, $j=1,2,...,n$.

We recall that f admits a discontinuity of the first kind at a point x_0, when it admits at this point a right limit $f(x_0^+)$ and a left limit $f(x_0^-)$, such that $f(x_0^+) \neq f(x_0^-)$.

Definition 1.3.3. *We say that f is of class C^1 piecewise on $[a,b]$, if there exists a subdivision $\left\{\left[x_{j-1}, x_j\right[,\ j = 1,2,...,n\right\}$ of $[a,b]$ such that:*
1. f is of class C^1 on each interval $\left]x_{j-1}, x_j\right[$, $j = 1,2,...,n$.
2. f admits right-hand derivatives and left-hand derivatives at the points x_j, $j = 1,2,...,n$ which are distinct..

Definition 1.3.4. *(Dirichlet kernel) We call the Dirichlet kernel the function D_n dédefined by:*

$$D_n(x) = \begin{cases} \dfrac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{2\sin\left(\frac{\pi x}{2l}\right)}, & \text{if } x \neq 2ml,\ m \in \mathbb{Z} \\ n + \dfrac{1}{2}, & \text{if } x = 2ml,\ m \in \mathbb{Z}. \end{cases} \tag{1.38}$$

Proposition 1.3.2. *The Dirichlet kernel D_n has the following properties:*
1. D_n is an even and periodic function of period $2l$.
2. D_n is a continuous function on $\mathbb{R}$.
3. D_n can be represented by the formula:

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos\left(\frac{k\pi x}{l}\right),\ x \in \mathbb{R}, \tag{1.39}$$

and furthermore, we have:

$$\frac{1}{l}\int_0^l D_n(x)dx = \frac{1}{2}. \tag{1.40}$$

Proof. 1. D_n is an even function (obvious).
On the other hand, for any $x \neq 2ml$, $m \in \mathbb{Z}$, we have:

$$\begin{aligned} D_n(x+2l) &= \frac{\sin\left[\left(n+\frac{1}{2}\right)\frac{\pi x}{l} + (2n+1)\pi\right]}{2\sin\left(\frac{\pi x}{2l}+\pi\right)} \\ &= \frac{-\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{-2\sin\left(\frac{\pi x}{2l}\right)} = \frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{2\sin\left(\frac{\pi x}{2l}\right)} \\ &= D_n(x). \end{aligned} \tag{1.41}$$

The function D_n is therefore $2l$ -periodic.

2. It suffices to show that D_n is continuous at the point $x_0 = 0$. Indeed,

$$
\begin{aligned}
\lim_{x\to 0} D_n(x) &= \lim_{x\to 0} \frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{2\sin\left(\frac{\pi x}{2l}\right)} \\
&= \lim_{x\to 0} \frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{\left(n+\frac{1}{2}\right)\frac{\pi x}{l}} \times \frac{\frac{\pi x}{2l}}{\sin\left(\frac{\pi x}{2l}\right)} \times \frac{\left(n+\frac{1}{2}\right)\frac{\pi x}{l}}{2\times\frac{\pi x}{2l}} \\
&= n+\frac{1}{2} = D_n(0).
\end{aligned}
$$

By periodicity of the function D_n, we then deduce that D_n is continuous at all points $x = 2ml$, $m \in \mathbb{Z}$. Therefore D_n is continuous on $\mathbb{R}$.

3. For all $x \neq 2ml$, $m \in \mathbb{Z}$, we have:

$$\begin{aligned}
D_n(x) &= \frac{1}{2} + \sum_{k=1}^{n} \cos(k\frac{\pi x}{l}) = \frac{1}{2} + \Re\left(\sum_{k=1}^{n} \exp(ik\frac{\pi x}{l})\right) \\
&= \frac{1}{2} + \Re\left[\exp(i\frac{\pi x}{l})\left(\frac{1-\exp(in\frac{\pi x}{l})}{1-\exp(i\frac{\pi x}{l})}\right)\right] \\
&= \frac{1}{2} + \Re\left[\exp(i\frac{\pi x}{l})\left(\frac{1-\cos\left(\frac{n\pi x}{l}\right)-i\sin\left(\frac{n\pi x}{l}\right)}{1-\cos(\frac{\pi x}{l})-i\sin(\frac{\pi x}{l})}\right)\right] \\
&= \frac{1}{2} + \Re\left[\exp(i\frac{\pi x}{l})\left(\frac{2\sin^2(\frac{n\pi x}{2l})-2i\cos\left(\frac{n\pi x}{2l}\right)\sin\left(\frac{n\pi x}{2l}\right)}{2\sin^2\left(\frac{\pi x}{2l}\right)-2i\cos\left(\frac{\pi x}{2l}\right)\sin\left(\frac{\pi x}{2l}\right)}\right)\right] \\
&= \frac{1}{2} + \Re\left[\exp(i\frac{\pi x}{l})\left(\frac{\cos\left(\frac{n\pi x}{2l}\right)+i\sin\left(\frac{n\pi x}{2l}\right)}{\cos\left(\frac{n\pi x}{2l}\right)+i\sin\left(\frac{n\pi x}{2l}\right)}\right)\left(\frac{\sin\left(\frac{n\pi x}{2l}\right)}{\sin\left(\frac{n\pi x}{2l}\right)}\right)\right] \\
&= \frac{1}{2} + \Re\left[\exp\left(i\left(\frac{n+1}{2}\right)\frac{\pi x}{l}\right)\left(\frac{\sin(\frac{n\pi x}{2l})}{\sin(\frac{n\pi x}{2l})}\right)\right] \\
&= \frac{1}{2} + \cos\left(\left(\frac{n+1}{2}\right)\frac{\pi x}{l}\right)\left(\frac{\sin\left(\frac{n\pi x}{2l}\right)}{\sin\left(\frac{n\pi x}{2l}\right)}\right) \\
&= \frac{1}{2} + \frac{1}{2}\left[\frac{\sin\left(\left(\frac{2n+1}{2}\right)\frac{\pi x}{l}\right)-\sin\left(\frac{\pi x}{2l}\right)}{\sin\left(\frac{\pi x}{2l}\right)}\right] \\
&= \frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{\sin\left(\frac{\pi x}{2l}\right)} = D_n(x).
\end{aligned} \tag{1.42}$$

If $x = 2ml$, $m \in \mathbb{Z}$, we have:

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos(2km\pi) = \frac{1}{2} + n. \tag{1.43}$$

In addition, we have:

$$\frac{1}{l}\int_0^l D_n(x)dx = \frac{1}{2l}\int_0^l dx + \frac{1}{l}\sum_{k=1}^{n}\int_0^l \cos(\frac{k\pi x}{l})dx = \frac{1}{2}. \tag{1.44}$$

□

Here is now the Dirichlet theorem:

Theorem 1.3.1. *(Dirichlet theorem) Let f be a $2l-$ periodic function of class C^1 piecewise on $\mathbb{R}$, then the Fourier series of f converges simply at every point x of $\mathbb{R} - 2l\mathbb{Z}$ and has the sum:*

$$S(x) = \frac{f(x^+) + f(x^-)}{2}. \tag{1.45}$$

Moreover, if f is continuous, the Fourier series of f converges uniformly on $\mathbb{R}$ and $S(x) = f(x)$, for all $x \in \mathbb{R}$.

Proof. Let's consider for all $n \in \mathbb{N}$, the sequence of partial sums of the Fourier series:

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l}). \tag{1.46}$$

Let x be any point of $\mathbb{R}$. It is enough to demonstrate:

$$\lim_{n\to+\infty} S_n(x) = \frac{f(x^+) + f(x^-)}{2}. \tag{1.47}$$

Indeed:

$$\begin{aligned}
S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \\
&= \frac{1}{2l}\int_{-l}^{l} f(t)dt + \sum_{k=1}^{n}\int_{-l}^{l} f(t)\left(\cos\left(\frac{n\pi t}{l}\right)\cos\left(\frac{n\pi x}{l}\right) + \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\right)dt \\
&= \frac{1}{l}\int_{-l}^{l} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos\left(\frac{n\pi (t-x)}{l}\right)\right)dt \\
&= \frac{1}{l}\int_{-l}^{l} f(t)D_n(t-x)\,dt \\
&= \frac{1}{l}\int_{-l-x}^{l-x} f(x+y)D_n(y)\,dy \quad \text{(using change } y = t-x\text{)} \\
&= \frac{1}{l}\int_{-l}^{l} f(x+y)D_n(y)\,dy \quad \text{(By Proposition 1.1.2)} \\
&= \frac{1}{l}\left(\int_{-l}^{0} f(x+y)D_n(y)\,dy + \int_{0}^{l} f(x+y)D_n(y)\,dy\right) \\
&= \frac{1}{l}\int_{0}^{l}\left(f(x-y)+f(x+y)\right)D_n(y)\,dy \text{ (by making the change } u \mapsto -u\text{).}
\end{aligned} \tag{1.48}$$

The use of formulas $\frac{1}{l}\int_0^l D_n(y)dy = \frac{1}{l}\int_0^l \left(\frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{l}\right)}{2\sin\left(\frac{\pi y}{2l}\right)}\right)dy = \frac{1}{2}$, we get:

$$\begin{aligned}
S_n(x) - \frac{f(x^+)+f(x^-)}{2} &= \frac{1}{l}\int_0^l \left(f(x-y)+f(x+y)\right)\left(\frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{l}\right)}{2\sin\left(\frac{\pi y}{2l}\right)}\right)dy + \\
&\quad -\left(\frac{f(x^+)+f(x^-)}{2l}\right)\int_0^l \left(\frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{l}\right)}{\sin\left(\frac{\pi y}{2l}\right)}\right)dy. \\
&= \frac{1}{2\pi}\int_0^l \frac{f(x-y)-f(x^-)}{y} \times \frac{\frac{\pi y}{l}}{\sin\left(\frac{\pi y}{2l}\right)} \times \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{l}\right)dy + \\
&\quad + \frac{1}{2\pi}\int_0^l \frac{f(x+y)-f(x^+)}{y} \times \frac{\frac{\pi y}{l}}{\sin\left(\frac{\pi y}{2l}\right)} \times \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{l}\right)dy.
\end{aligned}$$

Since the function f is of class C^1 piecewise and $\lim \dfrac{\frac{\pi y}{2l}}{\sin\left(\frac{\pi y}{2l}\right)} = 1$, the functions:

$$y \mapsto \frac{f(x-y)-f(x^-)}{y} \times \frac{\frac{\pi y}{l}}{\sin\left(\frac{\pi y}{2l}\right)} \text{ and } y \mapsto \frac{f(x+y)-f(x^+)}{y} \times \frac{\frac{\pi y}{l}}{\sin\left(\frac{\pi y}{2l}\right)}, \tag{1.49}$$

are bounded.

The result is therefore a consequence of the Riemann-Lebesgue lemma. □

1.3.4 Parseval's formula

Theorem 1.3.2. *Let f be a function defined on $\mathbb{R}$, periodic of period $2l$, integrable on $[-l, l]$ and let*

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right), \tag{1.50}$$

Then Parseval's formula is given by:

$$\frac{1}{l}\int_{-l}^{l} f^2(x)dx = \frac{a_0^2}{2} + \sum_{n\geq 1}\left(a_n^2 + b_n^2\right). \tag{1.51}$$

Proof. We demonstrate this theorem when

$$\sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right), \tag{1.52}$$

is uniformly convergent. Let us then assume that the numerical series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} b_n$ are absolutely convergent, and let (S_n) be the sequence of partial sums of the Fourier series such that

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\frac{k\pi x}{l}\right)\right). \tag{1.53}$$

For all $x \in \mathbb{R}$, we have:

$$\left|f(x) - S_n(x)\right| \leq \sum_{k\geq n+1} |a_k + b_k|, \tag{1.54}$$

quantity tending towards 0, when $n \to +\infty$. We deduce:

$$\begin{aligned} \left|f^2(x) - S_n^2(x)\right| &= \left|f(x) - S_n(x)\right|\left|f(x) + S_n(x)\right| \\ &\leq \left|f(x) - S_n(x)\right| \times 2\left|f(x)\right| \\ &\leq 2\sum_{k\geq n+1}|a_k + b_k|\left(\left|\frac{a_0}{2}\right| + \sum_{k\geq 1}|a_k + b_k|\right) \to 0, \text{ when } n \to +\infty. \end{aligned} \tag{1.55}$$

The sequence of functions S_n^2 is therefore uniformly convergent towards f^2. This uniform convergence allows us to write:

$$\frac{1}{l}\int_{-l}^{l} f^2(x)dx = \lim_{n\to+\infty}\left(\frac{1}{l}\int_{-l}^{l} S_n^2(x)dx\right). \tag{1.56}$$

Using the properties (1.18), (1.19), (1.20) and the fact that:

$$\left(\sum_{i=1}^{n}\alpha_i\right)^2 = \sum_{i=1}^{n}\alpha_i^2 + 2\sum_{i,j=1_{i\neq j}}^{n}\alpha_i\alpha_j, \tag{1.57}$$

$$\frac{1}{l}\int_{-l}^{l}\frac{a_0^2}{4}dx = \frac{a_0^2}{2}, \tag{1.58}$$

we can clearly deduce that

$$\frac{1}{l}\int_{-l}^{l} S_n^2(x)dx = \frac{a_0^2}{2} + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right), \tag{1.59}$$

By making n tend to $+\infty$, the expression (1.56), becomes

$$\frac{1}{l}\int_{-l}^{l} f^2(x)dx = \frac{a_0^2}{2} + \sum_{n\geq 1}\left(a_n^2 + b_n^2\right). \tag{1.60}$$

□

Remak 1.1. *The preceding Parseval theorem remains true even if f is not the sum of its Fourier series.*

1.4 Some applications of Fourier series

The use of the continuity of the sum of the Fourier series on certain intervals and Parseval's theorem gives us remarkable new identities:

Indeed, in the example 1.3.1, the Fourier series of x is given by:

$$SF(x) = 2\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sin(nx). \tag{1.61}$$

$x_0 = \frac{\pi}{2}$ is a point of continuity, we then have $\frac{\pi}{2} = 2\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sin\left(n\frac{\pi}{2}\right)$, so

$$\sum_{n\geq 0} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \tag{1.62}$$

On the other hand, using Parseval's theorem gives us

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{3}\pi^2 = 4\sum_{n\geq 1} \frac{1}{n^2}. \tag{1.63}$$

So:

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{1.64}$$

Similarly, in example 1.3.1, the Fourier series of x^2 is given by:

$$SF(x^2) = \frac{2}{3}\pi^2 + 4\sum_{n\geq 1} \frac{(-1)^n}{n^2} \cos(nx). \tag{1.65}$$

$x_0 = 0$ is a point of continuity, we then have $0 = \frac{2}{3}\pi^2 + 4\sum_{n\geq 1} \frac{(-1)^n}{n^2}$, hence

$$\sum_{n\geq 1} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{6} \tag{1.66}$$

On the other hand, using Parseval's theorem gives us:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{2}{5}\pi^4 = \left(\frac{2}{9}\pi^4 + 16\sum_{n\geq 1} \frac{1}{n^4}\right), \tag{1.67}$$

So:

$$\sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{1.68}$$

Bibliography

[1] J. Lelong Ferrand. Exercices résolus d'analyse. *Edition Dunod*, (1977).

[2] J. Lelong-Ferrand et J. M. Arnaudiés. Cours de mathématiques. *Tome 2, Edition Dunod*, (1978).

[3] J. Rivaud. Analyse Séries, équations différentielles: Exercices avec solutions. *Vuibert*, (1981).

[4] C. Servien. Analyse 3: Séries numériques, suites et séries de fonctions, Intégrales. *Ellipses*, (1995).

[5] J.P. Ramis et A. Warusfel. Mathématiques. Tout-en-un pour la Licence. *Niveau L1 Editions Dunod*.

[6] J. Dixmier. Cours de Mathématiques du premier cycle. *Editions Gauthier-Villars*.

[7] L. Bourguet. Sur les intégrales Eulériennes et quelques autres fonctions uniformes. *Acta Math.* 2, 261-295, 1883.