

## Series 3

**Exercise 1.** Specify, according to the value of the real number  $x$ , whether the following sentences define assertions, true assertions (i.e., propositions), or false assertions.

(I)

$$\cos^2(\tan x) \geq 0$$

(II)

$$\frac{1}{1+x} \leq 1$$

(III) The restriction of the sine function to the interval  $[0, x]$  is injective.

**Solution 1. (I)** The expression  $\cos^2(\tan x)$  is defined and non-negative for all  $x$  where  $\tan x$  is defined, i.e.,  $x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$ . - If  $x = \frac{\pi}{2} + k\pi$ , the sentence is not an assertion. - Otherwise, it is a **true proposition**.

**(II)** The expression  $\frac{1}{1+x}$  is defined for  $x \neq -1$ . - If  $x = -1$ , it is not an assertion. - For  $x \in (-\infty, -1) \cup [0, +\infty)$ , it is a **true proposition**. - For  $x \in (-1, 0)$ , it is a **false assertion**.

**(III)** The sine function is injective on intervals where it is strictly monotonic. - For  $x \in (0, \frac{\pi}{2}]$ , it is strictly increasing  $\Rightarrow$  **true proposition**. - For  $x > \frac{\pi}{2}$ , it is not injective  $\Rightarrow$  **false assertion**. - For  $x \leq 0$ , the interval  $[0, x]$  is not valid  $\Rightarrow$  **not an assertion**.

**Exercise 2.** Prove, without using truth tables, the following logical rules (valid for all propositions  $P$  and  $Q$ ):

(i)  $(P \vee Q) \iff (\neg P \Rightarrow Q)$  (ii)  $P \iff ((Q \Rightarrow P) \wedge (\neg Q \Rightarrow P))$  (iii)  $P \iff ((\neg P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q))$

**Solution 2. (i)** We transform the right-hand side:

$$\neg P \Rightarrow Q \iff \neg(\neg P) \vee Q \iff P \vee Q$$

Thus,  $(P \vee Q) \iff (\neg P \Rightarrow Q)$ .

**(ii)** Transform the right-hand side:

$$(Q \Rightarrow P) \wedge (\neg Q \Rightarrow P) \iff (\neg Q \vee P) \wedge (Q \vee P) \iff (P \vee \neg Q) \wedge (P \vee Q) \iff P \vee (\neg Q \wedge Q) \iff P$$

Thus,  $P \iff ((Q \Rightarrow P) \wedge (\neg Q \Rightarrow P))$ .

**(iii)** Transform the right-hand side:

$$(\neg P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q) \iff (P \vee Q) \wedge (P \vee \neg Q) \iff P \vee (Q \wedge \neg Q) \iff P \vee \text{False} \iff P$$

Thus,  $P \iff ((\neg P \Rightarrow Q) \wedge (\neg P \Rightarrow \neg Q))$ .

**Exercise 3.** Give the negation of the following five assertions:

1.  $P \wedge \neg Q$
2.  $P \vee (Q \wedge R)$
3.  $P \Leftrightarrow Q$
4.  $P \Rightarrow \neg Q$
5.  $P \Rightarrow (Q \Rightarrow R)$

**Solution 3.** Using logical rules:

1.  $\neg(P \wedge \neg Q) \iff \neg P \vee Q$
2.  $\neg(P \vee (Q \wedge R)) \iff \neg P \wedge (\neg Q \vee \neg R)$
3.  $\neg(P \Leftrightarrow Q) \iff (P \wedge \neg Q) \vee (Q \wedge \neg P)$
4.  $\neg(P \Rightarrow \neg Q) \iff P \wedge Q$
5.  $\neg(P \Rightarrow (Q \Rightarrow R)) \iff P \wedge Q \wedge \neg R$

**Exercise 4.** Analyze the structure of the following "assertions", consider possible variable changes, and replace commas with parentheses.

- (i)  $\forall x, (x = 0 \text{ or } y \neq 0) \Rightarrow \exists z, x = yz$  (ii)  $(x = 0 \text{ or } y \neq 0) \Rightarrow (\exists z, x = yz)$  (iii)  $(\forall x, \neg(x < y) \iff (x = y \text{ or } y < x))$  and  $(\forall z, z \geq x)$  (iv)  $\forall(x, y), x + y = 4 \implies \exists z, (x < z \text{ and } z < y)$

**Solution 4.** 1.  $\forall x((x = 0 \vee y \neq 0) \Rightarrow \exists z(x = y \cdot z))$

2.  $((x = 0 \vee y \neq 0) \Rightarrow (\exists z(x = y \cdot z)))$
3.  $(\forall x(\neg(x < y) \iff (x = y \vee y < x))) \wedge (\forall z(z \geq x))$
4.  $\forall x \forall y(x + y = 4 \Rightarrow \exists z(x < z \wedge z < y))$

**Exercise 5** (Proposition to DNF and CNF). Find the DNF and CNF for the proposition:  $(P \Rightarrow Q) \wedge (Q \Rightarrow R)$

**Solution 5. Step 1: Simplify the proposition**

$$(P \Rightarrow Q) \wedge (Q \Rightarrow R) \equiv (\neg P \vee Q) \wedge (\neg Q \vee R)$$

**Step 2: Find DNF**

We distribute to get DNF:

$$\begin{aligned} (\neg P \vee Q) \wedge (\neg Q \vee R) &\equiv [\neg P \wedge (\neg Q \vee R)] \vee [Q \wedge (\neg Q \vee R)] \\ &\equiv (\neg P \wedge \neg Q) \vee (\neg P \wedge R) \vee (Q \wedge \neg Q) \vee (Q \wedge R) \\ &\equiv (\neg P \wedge \neg Q) \vee (\neg P \wedge R) \vee (Q \wedge R) \quad (\text{since } Q \wedge \neg Q \equiv \text{False}) \end{aligned}$$

$$DNF = (\neg P \wedge \neg Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$$

**Step 3: Find CNF**

The expression is already in CNF:

$$CNF = (\neg P \vee Q) \wedge (\neg Q \vee R)$$

**Exercise 6** (Using Logical Equivalences). Convert the following proposition to both DNF and CNF using logical equivalences:

$$(P \wedge Q) \vee (\neg P \wedge R)$$

**Solution 6. Step 1: The expression is already in DNF**

$$DNF = (P \wedge Q) \vee (\neg P \wedge R)$$

**Step 2: Convert to CNF**

We use distribution:

$$\begin{aligned} (P \wedge Q) \vee (\neg P \wedge R) &\equiv [P \vee (\neg P \wedge R)] \wedge [Q \vee (\neg P \wedge R)] \\ &\equiv [(P \vee \neg P) \wedge (P \vee R)] \wedge [(Q \vee \neg P) \wedge (Q \vee R)] \\ &\equiv [True \wedge (P \vee R)] \wedge [(Q \vee \neg P) \wedge (Q \vee R)] \\ &\equiv (P \vee R) \wedge (Q \vee \neg P) \wedge (Q \vee R) \end{aligned}$$

$$CNF = (P \vee R) \wedge (\neg P \vee Q) \wedge (Q \vee R)$$

**Exercise 7** (Complex Proposition). Find the DNF and CNF for:  $(P \Leftrightarrow Q) \Rightarrow R$

**Solution 7. Step 1: Simplify the proposition**

$$\begin{aligned} (P \Leftrightarrow Q) \Rightarrow R &\equiv \neg(P \Leftrightarrow Q) \vee R \\ &\equiv \neg[(P \Rightarrow Q) \wedge (Q \Rightarrow P)] \vee R \\ &\equiv \neg[(\neg P \vee Q) \wedge (\neg Q \vee P)] \vee R \\ &\equiv [\neg(\neg P \vee Q) \vee \neg(\neg Q \vee P)] \vee R \\ &\equiv [(P \wedge \neg Q) \vee (Q \wedge \neg P)] \vee R \end{aligned}$$

**Step 2: DNF**

The expression is already in DNF:

$$DNF = (P \wedge \neg Q) \vee (Q \wedge \neg P) \vee R$$

**Step 3: CNF**

We need to convert to CNF. Let's start from the simplified form:

$$(P \wedge \neg Q) \vee (Q \wedge \neg P) \vee R$$

Using distribution:

$$\begin{aligned} &\equiv [((P \wedge \neg Q) \vee R) \vee (Q \wedge \neg P)] \\ &\equiv [(P \vee R) \wedge (\neg Q \vee R)] \vee (Q \wedge \neg P) \\ &\equiv [((P \vee R) \wedge (\neg Q \vee R)) \vee Q] \wedge [((P \vee R) \wedge (\neg Q \vee R)) \vee \neg P] \\ &\equiv [(P \vee R \vee Q) \wedge (\neg Q \vee R \vee Q)] \wedge [(P \vee R \vee \neg P) \wedge (\neg Q \vee R \vee \neg P)] \\ &\equiv [(P \vee Q \vee R) \wedge True] \wedge [True \wedge (\neg P \vee \neg Q \vee R)] \\ &\equiv (P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R) \end{aligned}$$

$$CNF = (P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R)$$

**Exercise 8 (Minimal Forms).** Find the minimal DNF and CNF for:  $(P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q)$

**Solution 8. Step 1: Simplify the CNF**

$$(P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q)$$

Using distribution and absorption:

$$(P \vee Q) \wedge (P \vee \neg Q) \equiv P \vee (Q \wedge \neg Q) \equiv P \vee False \equiv P$$

So the expression simplifies to:

$$P \wedge (\neg P \vee Q) \equiv (P \wedge \neg P) \vee (P \wedge Q) \equiv False \vee (P \wedge Q) \equiv P \wedge Q$$

**Step 2: Minimal DNF and CNF**

Since the expression simplifies to  $P \wedge Q$ :

- Minimal DNF:  $P \wedge Q$  - Minimal CNF:  $(P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q)$

However, the CNF can also be simplified to  $(P \vee Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q)$ , but the original form is already minimal in terms of the number of clauses.

**Exercise 9** (Three-Variable Proposition). *Find the DNF and CNF for:  $\neg(P \wedge Q) \Rightarrow (R \vee \neg P)$*

**Solution 9. Step 1: Simplify the proposition**

$$\begin{aligned}\neg(P \wedge Q) \Rightarrow (R \vee \neg P) &\equiv \neg\neg(P \wedge Q) \vee (R \vee \neg P) \\ &\equiv (P \wedge Q) \vee R \vee \neg P\end{aligned}$$

**Step 2: DNF**

*The expression is already in DNF:*

$$DNF = (P \wedge Q) \vee R \vee \neg P$$

*We can simplify further since  $\neg P$  makes the other terms redundant in some cases, but this is a valid DNF.*

**Step 3: CNF**

*Since we have a disjunction, to get CNF we consider the cases where the expression is false. The expression is false only when all three terms are false: -  $P \wedge Q$  is false when  $\neg P \vee \neg Q$  -  $R$  is false when  $\neg R$  -  $\neg P$  is false when  $P$*

*So the CNF is the conjunction of conditions that make the expression true:*

$$\begin{aligned}CNF &= (P \vee \neg P \vee R) \wedge (\neg Q \vee \neg P \vee R) \\ &\equiv True \wedge (\neg P \vee \neg Q \vee R) \\ &\equiv \neg P \vee \neg Q \vee R\end{aligned}$$

$$CNF = \neg P \vee \neg Q \vee R$$

## Summary

These exercises demonstrate various techniques for converting propositions to DNF and CNF:

1. **From truth tables:** Identify minterms for DNF and maxterms for CNF

2. **Using logical equivalences:** Apply distribution, De Morgan's laws, and simplification

3. **Direct conversion:** For simple propositions, read DNF/CNF directly

4. **Verification:** Use truth tables to confirm equivalence

Both DNF and CNF are useful in different contexts: DNF for circuit design and CNF for automated theorem proving.