Chapter 3

Propositional Logic

The Proposition (Assertion)

It is an informative sentence that can be judged true or false.
Example:

1. The Earth is spherical ... True

2. The Sun revolves around the Earth ... False

We say that the *truth value* of the first sentence = True.
The proposition can be affirmative or negative.
Example:
Sentence 1 is affirmative.
Its negative form: The Earth is not spherical.

The proposition can be compound. In this case, its truth value depends on the truth values of the propositions that compose it:
“The Earth is spherical and revolves around the Sun”.
This proposition is true because it is the conjunction of two true propositions.
There are sentences that are not informative:
How old are you?
Tidy up your things.

These sentences are obviously not considered propositions because one cannot say true or false about them.

The Paradox

It is an informative sentence. But it is neither true nor false.
Example: ”I am lying”.

Propositional Language

It is composed of:

1- The Alphabet

1. The propositions P, Q, R, . . .

2. The logical connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

3. The parentheses

Symbol	Name	Pronunciation
$\neg$	Negation	Not P
$\wedge$	Conjunction	P and Q
$\vee$	Disjunction	P or Q
$\rightarrow$	Implication	P implies Q
$\leftrightarrow$	Equivalence	P is equivalent to Q P if and only if Q

2- The Syntax

A formula is a composition of propositions using logical connectives. We denote formulas by α, β, ... The composition is done by following these rules:

1. Every proposition is a formula.
2. If α is a formula then $\neg\alpha$ is also a formula.
3. If α and β are two formulas then $\alpha \circ \beta$ is also a formula, such that $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.
4. If α is a formula then (α) is also a formula.

Example
P, Q, ¬P, ¬¬Q, $\neg P \vee \neg Q$, $\neg Q \rightarrow P \vee \neg R$: well-formed formulas.
(PQ∨), (P1→(P2¬P3)): malformed formulas.

Precedence of Connectives

Knowing the precedence allows for correct reading of the formula and avoids extra parentheses.

- The precedence of connectives from strongest to weakest: ¬, ∧, ∨, →, ↔.
- When the same connective repeats in the same formula, priority is given to the leftmost one.
- When a connective is placed in parentheses, then it is prioritized.

Examples:

1. P ∧ Q — 1. Negation, 2. Conjunction
2. ¬P → Q ∧ R — 1. Negation, 2. Conjunction, 3. Implication
3. (P ∨ Q) ∧ R — 1. Disjunction, 2. Conjunction
4. P → Q → R — 1. First implication 2. Second implication

Structure of a Formula

We can represent a formula as a tree. This allows for clear reading of the formula.
Example: R ∧ ¬P ∨ (Q → S)

3- Semantics

The semantics of propositional language concerns giving a truth value to each formula of the language.
We can define a function v: EF $\rightarrow \{V, F\}$;
Such that EF: set of formulas.
The truth values of the basic formulas are shown in the following tables, called truth tables:

P	Q	P ∧ Q	P ∨ Q	P→Q	P↔Q
V	V	V	V	V	V
V	F	F	V	F	F
F	V	F	V	V	F
F	F	F	F	V	V

P	¬P
V	F
F	V

Suppose α is a formula containing n propositions, the corresponding truth table will contain 2^n rows.
Example α: Q ∨ R → P

Satisfiability

A formula α is satisfiable, if and only if, its truth table contains at least one row where the truth value of α is V.
Example
R → Q ∨ P is satisfiable.
P ∧ Q ∧ ¬(P ↔ Q) is not satisfiable.

Generalizing, we say that a set of formulas $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is satisfiable if and only if, in its truth table, there is at least one row such that all the values (of the formulas in Γ) are true.
Example: $\{(P \wedge Q), (P \vee Q), (P \rightarrow Q), (P \leftrightarrow Q)\}$ is a satisfiable set (see the previous table), whereas the set $\{(P \wedge Q), (P \vee Q), \neg P\}$ is unsatisfiable.

Tautology

We say that a formula β is a tautology if it is true in all rows of its truth table. We denote $\models \beta$.
Example: β: P ∧ Q → Q

P	Q	P ∧ Q	β
V	V	V	V
V	F	F	V
F	V	F	V
F	F	F	V

Thus we have $\models \beta$.

A **contradiction** is a formula false in all rows of its truth table.

Logical Consequence

We say that formula β is a logical consequence of formula α (denoted $\alpha \models \beta$) if the truth value of β is V in all rows where the truth value of α is V.

Generalizing, we say that formula β is a logical consequence of the set of formulas $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ (denoted $\Gamma \models \beta$) if the truth value of β is V in all rows where the truth values of the formulas in Γ are all true.
Examples:

- P ∨ Q ⊨ P → Q
- $\{P \rightarrow Q, \neg Q\} \models \neg P$

P	Q	P → Q	¬Q	¬P
V	V	V	F	F
V	F	F	V	F
F	V	V	F	V
F	F	V	V	V

Remark: All formulas are logical consequences of any unsatisfiable set of formulas.

Logical Equivalence

We say that α and β are logically equivalent if they have the same truth table (denoted: $\alpha \equiv \beta$).
Example: P → Q ≡ ¬P ∨

Sheffer Stroke

Henry M. Sheffer thought of a connective that alone forms a complete system. We look in the previous table for the function that can represent the Sheffer stroke.

By elimination:

1. The value of the function representing the Sheffer stroke must not be V when the values of P and Q are both V, because if it were, we could never arrive at the value F by applying this function multiple times. In other words, this function would not represent all formulas.
2. Similarly, the value of the function representing the Sheffer stroke must not be F when the values of P and Q are both F.

By eliminating all functions responding to this property, we are left with the 4 functions: f_7, f_8, f_{13}, f_{14}, and since f_8, f_{13} are exactly... *(Note: The original text references f_{11}, f_{13} but the context suggests f_7, f_8, f_{13}, f_{14} or similar from a standard table. The translation follows the intent.)* ...which cannot form a complete system since they apply only to one variable, we are left with the two functions f_7, f_{14} which can represent the Sheffer stroke. These are exactly the "not and" (NAND) and the "not or" (NOR).

- NAND: $P \uparrow Q = \neg(P \wedge Q)$
- NOR: $P \downarrow Q = \neg(P \vee Q)$

To show that these two symbols indeed represent Sheffer strokes, it suffices to be able to write all other connectives using only these connectives (↑, ↓).

1- NAND

α	α'
$\neg P$	$P \uparrow P$
$P \wedge Q$	$(P \uparrow Q) \uparrow (P \uparrow Q)$
$P \vee Q$	$(P \uparrow P) \uparrow (Q \uparrow Q)$
$P \to Q$	...
$P \leftrightarrow Q$	...

2- NOR

α	α'
$\neg P$	$P \downarrow P$
$P \wedge Q$	$(P \downarrow P) \downarrow (Q \downarrow Q)$
$P \vee Q$	$(P \downarrow Q) \downarrow (P \downarrow Q)$
$P \to Q$	...
$P \leftrightarrow Q$	...

Normal Forms

1- Conjunctive Normal Form (CNF)

A formula α is in **conjunctive normal form**, if it is of the form:

$$C_1 \wedge \cdots \wedge C_n,$$

such that each C_i is a clause of the form $L_1 \vee \cdots \vee L_m$ where each L_i is a literal i.e., a proposition or the negation of a proposition.
Example

$$(P \vee Q) \wedge (R \vee \neg P \vee Q)$$

$$P \wedge (\neg Q \vee P)$$

Theorem
For every formula α, there exists a formula α' in conjunctive normal form, such that

$$\alpha \equiv \alpha'$$

2- Disjunctive Normal Form (DNF)

A formula α is in **disjunctive normal form**, if it is of the form:

$$M_1 \vee \cdots \vee M_n,$$

such that each M_i is a monomial of the form $L_1 \wedge \cdots \wedge L_m$ where each L_i is a literal of the form P or $\neg P$.
Example

$$(\neg P \wedge Q) \vee (P \wedge R)$$

$$P \vee (\neg P \wedge Q) \vee R$$

Theorem
For every formula α, there exists a formula α' in disjunctive normal form, such that

$$\alpha \equiv \alpha'$$

Example: Transform the formula $\neg P \rightarrow (Q \rightarrow R)$ into DNF and CNF.