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(d) *Theorem of complementary energy (Engesser's energy theorem)*

This theorem can be stated as follows:

Of all the states satisfying the conditions of equilibrium in a body, the state of stress will be such that the complementary energy is a minimum.

The advantage of the complementary energy principle is that it can be applied to situations where non-linear relationship exists between load and deformation. Considering Fig. 10.39,

Strain energy, $U = \int P\delta e$, where P is the load corresponding to deformation δe.

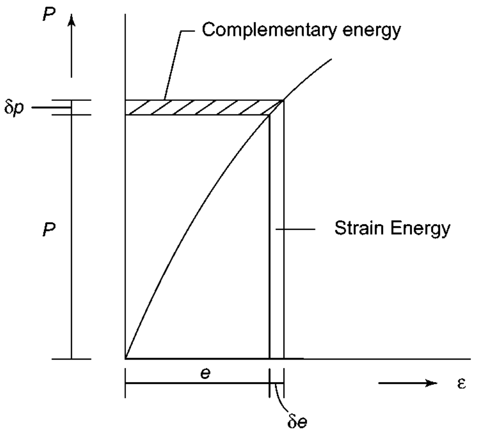


Fig. 10.39 Engesser's energy theorem

This is derived from the integration of the shaded area below the curve, which is the work done by the gradually increasing force *P*.

Above the curve is an area that represents a fictitious quantity called complementary energy. This area is given by the relation

$$U^* = \int e\delta P$$

The asterisk over U is to distinguish the complementary energy from the strain energy.

10.9.2 Castigliano's Theorems

Alberto Castigliano in 1873 described methods to find deflections and slopes in beams and trusses as his dissertation for engineering diploma, which are known as Castigliano's theorems. This is a classic example of application of energy principles to calculating deformations. The two conditions that govern his proposition are that the body is stressed within elastic limit and the deformations are linear functions of the loads.

Castigliano's first theorem is a special case of Engesser's theorem of complementary energy with the proviso that the deformations must be linear functions of the loads. The theorem can be stated as follows:

In a linearly elastic system, the partial derivative of the total strain energy stored in a structure with respect to the displacement at a point is equal to the force *at that point.*

Stated mathematically, $\delta U/\delta\Delta = P$

Considering a linearly elastic material, it is clear that the strain energy and complementary energy are equal. Thus, according to Engesser's theorem of complementary energy, $U^* = \int \Delta \delta P$; $U = \int P\delta\Delta$ and the two are equal in the case of linearly elastic material.

Castigliano's second theorem is again limited to linearly elastic systems. Stated simply,

In a linearly elastic system, the partial derivative of the total strain energy stored in the structure with respect to any force gives the displacement at that point in the direction of the force.

In the above theorem, displacement can mean a translation/rotation and the force can be a load or a couple.

To prove the theorem, consider the beam loaded as shown in Fig. 10.40.

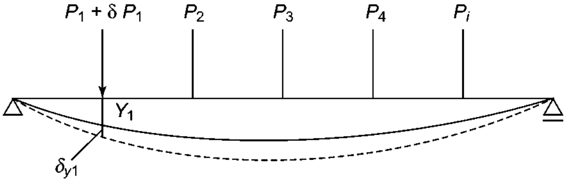


Fig. 10.40 Castigliano's theorems

$P_1, P_2, \ldots$ are the gradually applied loads acting on the beam and let $y_1, y_2, \ldots$ be the deflections under the loads. Then the total work done by the external forces is

$W_e = U = (1/2)P_1y_1 + (1/2)P_2y_2, \ldots$, where U is the strain energy stored in the beam.

Now we increase load P_1 to $P_1 + \delta P_1$ and let the additional deflections be δy_1, $\delta y_2, \ldots$, etc.

The additional work done by the loads can be expressed as

$\delta W_e = \delta U = (1/2)\delta P_1 ä\ y_1 + P_1\delta y_1 + P_2\delta y_2 + \ldots$

Total strain energy U_t stored with the two applications of the loads is

$U_t = [(1/2)P_1y_1 + (1/2)P_2y_2 + \ldots] + [(1/2)\delta P_1\delta y_1 + P_1\delta y_1 + P_2\delta y_2 + \ldots]$

Now consider the same beam loaded simultaneously with the same loads but $P_1 + \delta P_1$ applied along with P_2, P_3 gradually. The deflection under load $P_1 + \delta P_1$ will be $y_1 + \delta y_1$, under load $P_2 y_2 + \delta y_2, \ldots$, etc. The total external work done or strain energy stored can be expressed as

$U_t = (1/2)(P_1 + \delta P_1) + (1/2)P_2(y_2 + \delta y_2) + \ldots$

Since the strain energy must be the same due to the two methods of application of the load, the two strain energies obtained must be equal. Therefore,

$$[(1/2)P_1y_1 + (1/2)P_2y_2 + \ldots] + [(1/2)\delta P_1\delta y_1 + P_1\delta y_1 + P_2\delta y_2 + \ldots]$$
$$= (1/2)(P_1 + \delta P_1) + (1/2)P_2(y_2 + \delta y_2) + \ldots$$

Simplifying, we get

$(1/2)P_1\delta y_1 + (1/2)P_2\delta y_2 + \ldots = (1/2)\delta P_1 y_1$

We have neglected the term $\delta P_1 \ddot{a}\ \ y_1$ being a small quantity of second order. The left-hand side is the δU, the increase in strain energy calculated earlier.

We thus get $\delta U/\delta P = y_1$, the deflection under the load P_1.

Castigliano's theorem can thus be stated as follows:

The partial derivative of the total strain energy of any structure, which is linearly elastic, with respect to any of the applied forces is equal to the displacement of the point of application of that force in the direction of the force.

We have derived the theorem for a set of point loads. This is equally applicable for the moments applied on the structure. The displacement in this case will be the rotation of the point of application of the moment in the direction of the applied moment.

Castigliano's theorem gives deflection $y = \delta U/\delta P$. When we consider deflection due to bending, $U = \int M^2_x dx/2EI$, the integration being done over the segment or span length.

Therefore, $y = \delta[\int M^2_x dx/2EI]/\delta P$. The partial derivative is taken with respect to the load P, which gives the deflection in the direction of P. It will be convenient to take the derivative within the integral sign before integrating. Thus,

$$y = \int 2M(\delta M_x/\delta P)dx/2EI = S\ [M(\delta M/\delta P)dx/EI]$$

If the deflection is required at a point where no load is acting, a fictitious load can be assumed to be acting. This load is set to zero after differentiation but before integration.

The following examples illustrate the application of the theorem in calculating deflections and slopes.

Example 10.24 Castigliano's theorem: cantilever

A cantilever of span L carries a concentrated load W at the free end. Find the slope and deflection at the free end using Castigliano's theorem.

Solution The cantilever with the load is shown in Fig. 10.41.

Bending moment at x, $M_x = Px$

The strain energy stored in the beam due to flexure, neglecting shear strain energy, is

$$U = \int [M^2 dx/2EI] = \int [P^2x^2 dx/2EI]$$

$$dU/dP = d[\int [P^2x^2dx/2EI]/dP = \int [Px^2dx/EI]$$, differentiating within the integral sign

$$dU/dP = \text{deflection } y = PL^3/3EI$$, integrating from 0 to L

To find the slope at B, apply a fictitious couple μ at A as shown. The moment diagram due to this couple is uniform for the cantilever. Moment at x due to this couple = μ.

Total moment at $x = Px + \mu$

Strain energy, $U = \int (Px + \mu)^2 dx/2EI$

$$dU/d\mu = d[\int (Px + \mu)^2dx/2EI]/d\mu = \int [(Px + \mu)dx/EI]$$, differentiating within the integral

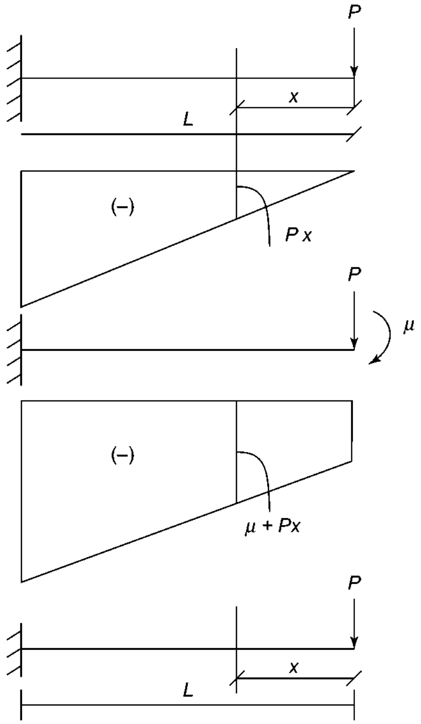


Fig. 10.41

We can now set the fictitious moment $\mu = 0$ and then integrate with respect to x.

$$dU/d\mu = \text{slope at } B,\ \theta_B = PL^2/2EI$$

Example 10.25 Castigliano's theorem: SS beam with UD load

A simply supported beam is carrying a uniformly distributed load w/m throughout its span. Find the deflection at the centre and the slopes at the ends of the beam.

Solution The beam with the loading is shown in Fig. 10.42.

BM at x from $A = wLx/2 - wx^2/2$

As there is no load at the mid-point of the beam, where the maximum deflection would be, we put a fictitious load P. This load gives the BM diagram shown. We take advantage of the symmetry of the loading and integrate for strain energy from 0 to $L/2$ and double that.

$$M_x = [wLx/2 - wx^2] + Px/2$$

$$\delta M_x/\delta P = x/2$$

$$\frac{\delta U}{\delta P} = \int M\,(\delta M/\delta P)dx/EI = 2\int_0^{L/2} [wLx/2 - wx^2/2 + Px/2](x/2)dx/EI$$

After differentiating, P can be set to zero before integrating.

$$\frac{\delta U}{\delta P} = 2\int [wLx/2 - wx^2/2](x/2)dx/EI$$

Integrating from 0 to $L/2$,

$$\delta U/\delta P = \text{deflection under } P = 2[wLx^3/12 - wx^4/16]_0^{L/2}/EI$$

Deflection under $P = y_c = (2/EI)[wL^4/96 - wL^4/256] = (5/384)wL^4/EI$

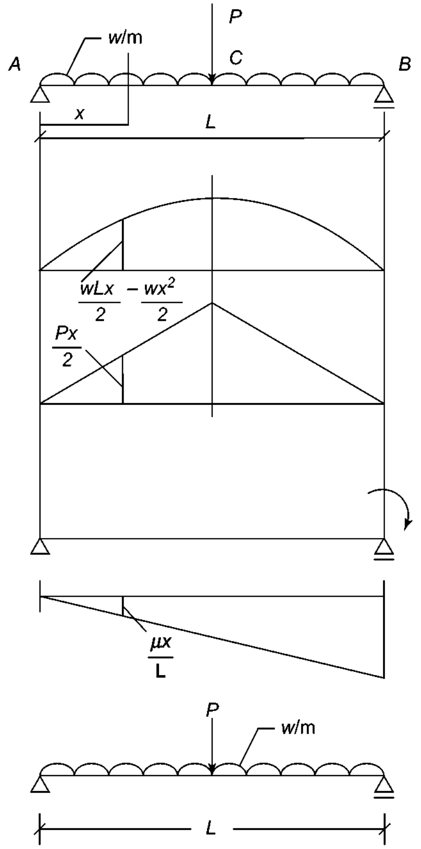


Fig. 10.42

Due to symmetry, slopes at A and B will be the same though of different signs. To find the slope at B, we apply a fictitious moment μ at B as shown. At x from A,

Moment due to $\mu = \mu x/L$

Total moment at x, $M_x = [wL\,x/2 - wx^2/2 + \mu x/L]$

$\delta M_x/\delta\mu = x/L$

$\delta U/\delta\mu$ = slope at B = $\int M(\delta M/\delta\mu)dx/EI$, integration being done from 0 to L

Slope at B, $\theta_B = \int[wLx/2 - wx^2/2 + \mu x/L](x/L)dx/EI$

We can now set the fictitious couple μ to zero.

$$\theta_B = \int[wL\,x/2 - wx^2/2]\,(x/L)dx/EI$$

$$= \frac{Wx^3/6}{EI} - \frac{Wx^4/8L}{0}$$

$$= wL^3/24EI$$

The sign being positive, the slope is in the direction of μ, i.e. anticlockwise.

Example 10.26 Castigliano's theorem: SS beam with point load

A simply supported beam is carrying a point load P at a from the left end. Find the deflection under the load, slopes at the ends.

Solution The beam with the loading is shown in Fig. 10.43.

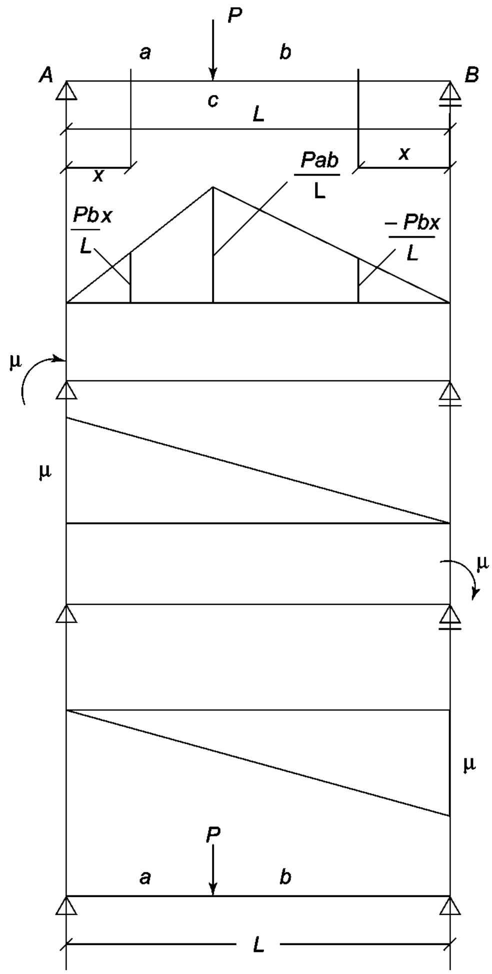


Fig. 10.43

The reaction at A is Pb/L and the reaction at B is Pa/L. As the bending equations are different in the two segments AC and CB, we have to calculate the strain energy in parts.

In segment AC, $M_x = Pbx/Lx = 0$ to $x = a$; $\quad \delta M_x/\delta P = bx/L$

In segment BC, $M_x = Pax/L$ (calculating from right), $x = 0$ at B and $x = b$ at C; $\delta M_x/\delta P = ax/L$

(a) *Deflection under the load* $\quad U = M_x^2 dx/EI$; $\delta U/\delta P = \int [M(\delta M/\delta P)dx/EI]$

$$\frac{\delta U}{\delta P} = \int_0^a (Pbx/L)(bx/L)dx/EI + \int_0^b (Pax/L)(ax/L)dx/EI$$

$$y = \frac{Pb^2}{EI\,L^2}\,[x^3/3]_0 + \frac{Pa^2}{EI\,L^2}\,[x^3/3]_0 = \frac{Pa^2b^2(a+b)}{3EIL^2} = \frac{Pa^2b^2}{3EIL}$$

(b) *Slope at A* For finding the slope at A, apply a fictitious moment μ at A as shown.

BM at x $M_x = Pb\,x/L - \mu(1 - x/L)$, $0 < x < a$ [origin at A]

$= Pa\,x/L + \mu\,x/L$, $0 < x < b$ [origin at B]

$\delta M_x/\delta\mu = 1 - x/L$, $0 < x < a$ and $\delta M_x/\delta\mu = x/L$; $0 < x < b$.

$\delta U/\delta\mu$ = slope at A, $\theta_A = \int M\,(\delta M_x/\delta\mu)\,dx/EI$

$$= \int_0^a [(Pb\,x/L + \mu(1 - x/L)\,(1 - x/L)\,dx/EI$$

$$+ \int_0^b [(Pa\,x/L + \mu x/L)\,(x/L)\,dx/EI$$

$$= \frac{1}{EI}\,[Pb\,x^2/2L - Pbx^3/3L^2\,]_0^a + \frac{1}{EI}\,[\,Pa\,x^3/\,3L^2]_0^b$$

$$= 1/EI\,[Pba^2/2L - Pba^3/3L^2 + Pab^3/3L^2]$$

$$= Pab\,\frac{[3aL - 2a^2 + 2b^2]}{6\,EI\,L^2}$$

$$= Pab\,\frac{[3aL - 2L\,(a - b]}{6\,EI\,L^2}$$

$$= Pab(L + b)/6EIL.$$

(c) *Slope at B* To find the slope at B, we apply a fictitious moment μ at B.

BM in segment $AC = Pbx/L - \mu x/L$

BM in segment $BC = Pax/L + m(1 - x\,/L)$

$\delta M/\delta\mu = -x/L$ in segment AC and $(1 - x/L)$ in segment BC

$$\delta U/\delta\mu = \int [Pbx/L - \mu x/L](x/L)dx/EI + \int [Pax/L + \mu(1 - x/L)](1 - x/L)dx/EI$$

The first integration is from 0 to a and the second from 0 to b. We set $\mu = 0$ before performing the integration. This will give θ_B.

$$EI\,\theta_B = \frac{Pba^3}{3L^2} + \frac{Pba^2}{2L} - \frac{Pba^3}{3L^2}$$

$$\theta_B = \frac{Pab(L + a)}{6EIL}$$

Example 10.27 Casstgliano's theorem: deflection in beam

In the beam loaded as shown in Fig. 10.44, find the vertical deflection and slope at the free end. EI is constant.

Solution To find the vertical deflection, we apply a fictitious load P at the free end. The moment M_x can be written for the segments AB and BC separately. Due to UD load,

Reaction at $B = (30 \times 10 \times 5)/8 = 187.5$ kN ↑

Reaction at $A = (30 \times 8 \times 4 - 30 \times 2 \times 1)/8 = 112.5$ kN ↑

Due to load P at C, reaction at $A = (-P \times 2)\,/8 = -\,P/4$ ↓

Reaction at $B = P \times 10\,/8 = 1.25\,P$ ↑

M_x in $AB = 112.5x - 30x^2/2 - Px/4 = 112.5x - 15x^2 - Px/4$

$\delta M_x/\delta P = -x/4$

M_x in $BC = -Px - 30x^2/2 = -P\,x - 15x^2$

$\delta M_x/\delta P = -x$

Deflection at C, $y_c = \delta U/\delta P = \int [M(\delta M_x/\delta P)dx/EI$

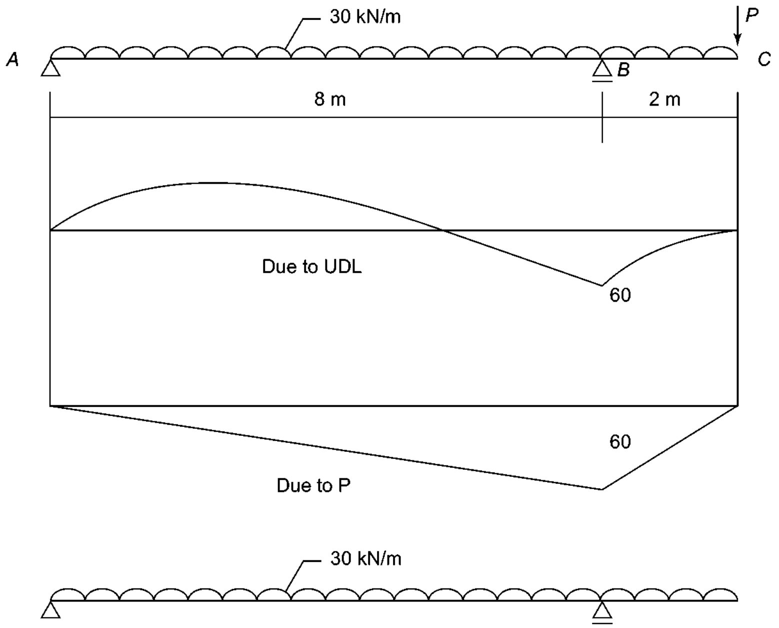


Fig. 10.44

$$y_c = \int_0^8 \left[112.5x - 15x^2 - \frac{Px}{4}\right]\left(-\frac{x}{4}\right)\frac{dx}{EI} + \int_0^2 \left[-(Px - 15x^2)\right](-x)\,\frac{dx}{EI}$$

Setting $P = 0$ as the load is fictitious,

$$y_c = \frac{1}{EI}\left[-\frac{112.5x^3}{24} + \frac{15x^4}{16}\right]_0^8 + \frac{1}{EI}\left[-\frac{15x^3}{3}\right]_0^2$$

$$= \frac{1}{EI}\left[-\frac{112.5(8)^3}{24} + \frac{15(8)^3}{16}\right] + \frac{1}{EI}[-5\times 8]$$

$$= \frac{1}{EI}[-2400 + 480] + \frac{-40}{EI} = \frac{-1960}{EI} \quad \text{(upward deflection)}$$

To find the slope at C, apply a fictitious clockwise moment μ at C.

The bending moment due to this couple at C is

$M_x = -\mu x/8$ in AB and $-\mu$ in BC (from the right)

$\delta M_x/\delta\mu = -(x/8)$ in AB and -1 in BC

Slope at C, $\theta_C = \int M(\delta M/\delta\mu)dx/EI$

$$y_c = \int_0^8 (112.5x - 15x^2 - \mu x/8)(-x/8)dx/EI + \int_0^2 (-15x^2 - \mu](-1)dx/EI$$

$$= \frac{-112.5x^3/16 + 15x^3/24]_0^8}{EI} + \frac{[+5x^3]_0^2}{EI} \quad \text{setting } \mu = 0 \text{ before integrating}$$

$= -2920/EI$ (anticlockwise slope)

Example 10.28 Castigliano's theorem: cantilever frame

The cantilever frame with rigid joint is loaded as shown in Fig. 10.45. Find the horizontal and vertical deflections at the free end.

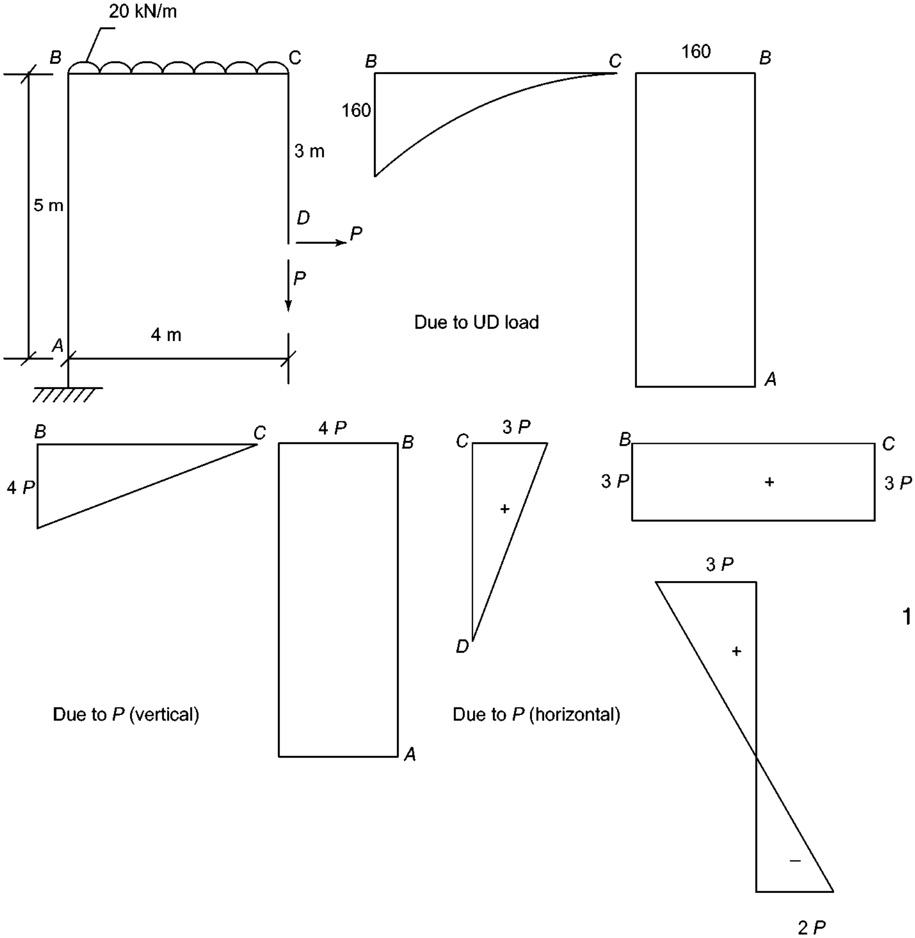


Fig. 10.45

Solution To find the vertical deflection at D, we apply a fictitious load P at the free end. In such cases, we consider the BM in each member of the frame separately. The moments due to the applied load and the BM due to P are as shown in Table 10.1.

Table 10.1

Segment	M_x	*Moment due to P*	$\Delta(M_x + M_P)\delta P$	*Origin at*	*Limits of integration*
DC	0	0	0	*D*	0 to 3
CB	$-10x^2$	$-Px$	$-x$	*C*	0 to 4
BA	-160	$-4P$	-4	*B*	0 to 5

Vertical deflection at $D = \delta U/\delta P = \Sigma M(\delta M/\delta P)dx/EI$

$$y_D = \int 0 + \int_0^4 (-10x^2 - Px)(-x)dx/EI + \int_0^5 (-160 - 4P)(-4)dx/EI$$

We set P to zero before integrating as P is a fictitious load.

$$= \frac{1}{EI}\,[10x^4/4]_0^4 + [640x]_0^5$$

$$= (640 + 3200)/EI = 3840/EI$$

To find the horizontal deflection at D, apply a fictitious load P horizontally at D. The moments in the segments are as shown in Table 10.2.

Table 10.2

Segment	M_x	M_P	$\Delta(M_x + M_P]/\delta P$	*Origin at*	*Limits of integration*
DC	0	$P\,x$	X	D	0 to 3
CB	$-10\,x^2$	$3P$	3	C	0 to 4
BA	-160	$3P - Px$	$3 - x$	B	0 to 5

The horizontal deflection at $D = h_D = \Sigma \int M(\delta M/\delta P)dx/EI$

$$h_D = \int_0^3 (0 + Px)(x)dx/EI + \int_0^4 (-10x^2 + 3P)(3)dx/EI$$

$$+ \int_0^5 (-160 + 3P + Px)(3 - x)dx/EI$$

$$= \left[\frac{-10x^3}{EI}\right]_0^4 + \left[\frac{-480x + 160x^2/2}{EI}\right]_0^5, \text{ setting } P \text{ to zero before integrating}$$

$$= -640 + (-2400) + 2000 = -1040/EI$$

The negative sign shows that the deflection is opposite to the direction of P assumed.

Example 10.29 Castigliano's theorem: curved bar

The beam in the shape of a quarter circle is loaded at the free end with a load of 30 kN. Find the horizontal and vertical deflections at the free end.

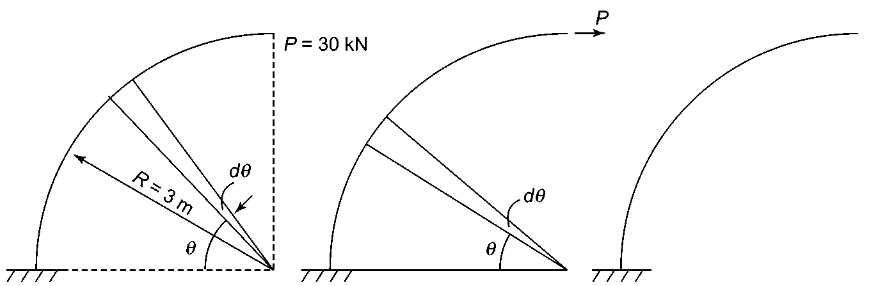


Fig. 10.46

Solution We consider an elementary strip of length ds at an angle θ from the base as shown. We have $ds = Rd\theta$, where R is the radius. The moment of the 30 kN load = $PR\cos\theta$.

As the applied load is in the direction of the deflection required, we find the strain energy as $\int M^2 ds/EI$.

$$U = \int_0^{\pi/2} [(PR\cos\theta)^2 Rd\theta/EI] = \frac{P^2R^3}{EI}\int_0^{\pi/2}\frac{(1+\cos 2\theta)}{2}$$

$$= \frac{P^2R^3}{2EI}[\theta + \sin 2\theta]_0^{\pi/2}$$

$$= P^2R^3\delta/4EI = P^2R^3\delta/4EI$$

$\delta U/\delta P$ = vertical deflection $= 2PR^3\pi/4EI = 2 \times 30 \times 3^3 \times (\delta)/4EI = 1272/EI$

To find the horizontal deflection, we apply a fictitious horizontal load as shown.

Moment due to 30 kN load = $30R\cos\theta$

Moment due to $P = PR(1 - \sin\theta)$

$\delta M/\delta P = R(1 - \sin\theta)$

$$\frac{\delta U}{\delta P} = \int_0^{\pi/2} [M(\delta M/\delta P)ds/EI]\text{, gives the horizontal deflection}$$

$$y_h = \int [30R\cos\theta + PR(1-\sin\theta)][R(1-\sin\theta)]Rd\theta/EI$$

Setting $P = 0$ and integrating,

$$= \int_0^{\pi/2}\frac{[30R^3\cos\theta]}{EI}[1 - \sin\theta]d\theta = \frac{30R^3}{EI}\int_0^{\pi/2}(\cos\theta - \cos\theta\sin\theta)d\theta$$

$$= \frac{30R^3}{EI}\left[\sin\theta + \frac{\cos 2\theta}{4}\right]_0^{\pi/2}$$

$= -30R^3/2EI = -405/EI$ (The negative sign shows that the displacement is against the direction of P assumed.)

Example 10.30 Castigliano's theorem: bent bar

In the bent bar shown in Fig. 10.47, find the deflection and slope at the free end.

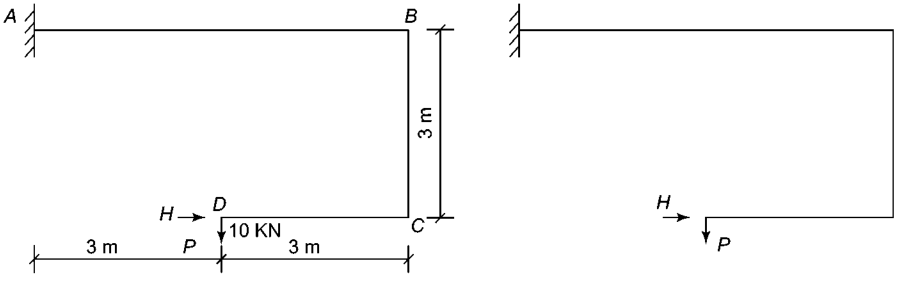


Fig. 10.47

Solution As different segments have different bending moment equations, we list down the moments as shown in Table 10.3. As the vertical deflection to be found is in the direction of the applied load, we take the load as P and substitute its value after integration.

Table 10.3

Segment	M_x	*Origin at*	*Limits of integration*
DC	Px	*D*	0 to 3 m
CB	$3P$	*C*	0 to 3 m
BA	$3P - Px$	*B*	0 to 6 m

Strain energy, $U = \int M^2 dx/2EI$

$$= \int_0^3 \frac{(Px)^2 dx}{2EI} + \int_0^3 \frac{(3P)^2 dx}{2EI} + \int_0^6 \frac{(3P - Px)^2 dx}{2EI}$$

$$= \frac{[P^2x^3]_0^3}{2EI} + \frac{[3Px]_0^3}{2EI} + \frac{[9P^2x + P^2x^2 - 6P^2x^3]_0^6}{2EI}$$

$$= 54P^2/2EI$$

Here $\delta U/\delta P$ gives the vertical deflection at D.

$\delta U/\delta P = 54P/EI = 540/EI$

Vertical deflection of $D = 540/EI$

To find the horizontal deflection, we apply a fictitious horizontal force H at D as shown. The bending moments in the different segments are listed as shown in Table 10.4.

Table 10.4

Segment	*Moment due to loads*	*Moment due to H*	*Origin at*	*Limits of integration*
DC	$10x$	0	*D*	0 to 3 m
CB	30	Hx	*C*	0 to 3 m
BA	$30 - 10x$	$3H$	*B*	0 to 6 m

Horizontal deflection is given by $\int M(\delta M/\delta H)dx/EI$.

For the segment DC, $\delta M/\delta H = 0$. The integration has to be done for the other two segments only.

$$\text{Horizontal deflection} = \int_0^3 [(30 + Hx)x dx/EI] + \int_0^6 [(30 - 10x + 3H)3dx/EI]$$

$$= \frac{[30x^2/2]_0^3}{EI} + \frac{[90x - 30x^2/2]_0^6}{EI}$$

$$= 135/EI$$

Example 10.31 Castigliano's theorem: tapering section

A cantilever beam, 2 m long, has uniform width but its depth is varying from 200 mm at the free end to 400 mm at the fixed end. If $E = 200$ GPa and the width is 100 mm, find the deflection at the free end due to point load of 40 kN acting at the free end.

Solution The beam is shown in Fig. 10.48.

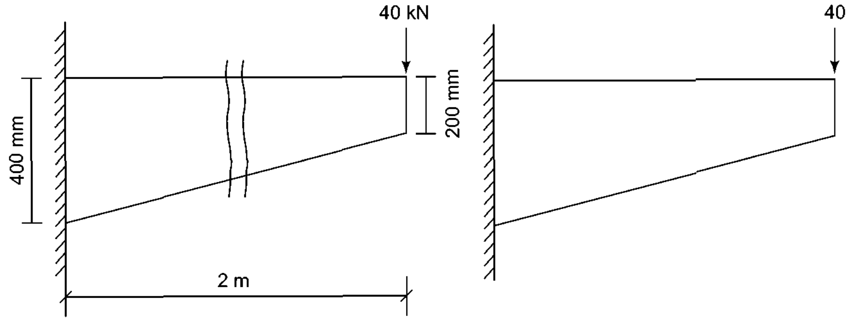


Fig. 10.48

As the depth is varying, *MI* changes with x. MI has to be expressed as a function of x.

Bending at x m from the free end = Px kNm

Depth of beam at x m = 200 + (400 – 200)x/2 = (200 + 100x) mm

MI at x, $I_x = 100(200 + 100x)^3/12 = 100^4(2 + x)^3/12$ mm^4

$= (2 + x)^3/(12 \times 10^4)$ m^4

Strain energy, $U = \int M^2 dx/EI$

$$= \int_0^2 \frac{P^2x^2(12 \times 10^4)dx}{E(2+x)^3}$$

$$= \frac{P^2 \times 12 \times 10^4}{E} \int_0^2 \frac{x^2 dx}{(2+x)^3}$$

To integrate this expression, we take

$y = 2 + x, x = y - 2, x^2 = y^2 - 4y + 4, dx = dy$

$$\int_0^4 \frac{(y^2 - 4y + 4)dy}{y^3} = \int_0^4 (y^{-1} - 4y^{-2} + 4y^{-3})dy = [\log_e y + 4/y - 4/2y^2]_2^4$$

$= [\log_e(2 + x) + 4/(2 + x) - 4/2(2 + x)^2]_0^2$

$= \log_e 4 - \log_e 2 + 1 - 2 - 32 + 1/2 = -31.8$

$U = P^2 \times 12 \times 10^4 (-31.8)/200 \times 10^6 = -0.019 \times 10^{-4} P^2$

$\delta U/\delta P = -0.038 \times 10^{-4} P$ m = 0.15 mm

Deflection at the free end = 0.15 mm

10.9.3 Unit Load Method

The unit load method, also known as the dummy load method, can be applied to beams and trusses. The unit load method is a versatile method and simple to apply. The application to beam deflections and slopes is illustrated here.

The unit load method can be directly derived from Castigliano's first theorem. We derive it from the first principles. Consider a body subjected to forces (and moments) as shown in Fig. 10.49. The deflection at point C is required where a unit load is applied.

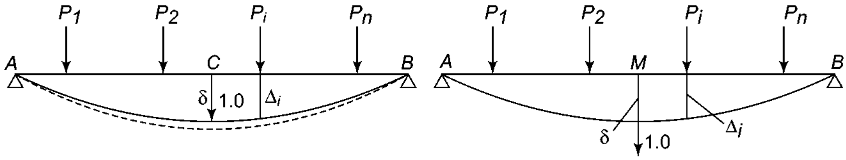


Fig. 10.49 Derivation of unit load method

The applied loads cause stresses in the body. If we consider an elemental area dA of length dx, the force in the element due to internal stresses is σdA. This force shortens or elongates the elemental length dx. Let the deformation of the element be dL. Then the elemental work done by the internal forces on the element is $(1/2)\sigma dA\,dL$. The total internal work, which is stored as internal strain energy, is $(1/2)\Sigma(\sigma dA\,dL)$, summing up the energies stored in all such elements. If the deflections due to the gradually applied loads P_1, P_2, ... are Δ_1, Δ_2, ..., then the external work done by the forces is $(1/2)P_1\Delta_1$, $(1/2)P_2\Delta_2$, and so on. The total external work is $(1/2)\Sigma P_i\Delta_i$.

By the law of conservation of energy,

$$(1/2)\Sigma p_i\Delta_i = (1/2)\Sigma(\sigma dA\,dL) \qquad (10.1)$$

If we apply a unit load before the loads P_i are applied, then the loads P_1, P_2, etc. are gradually applied and the external work and strain energy can be calculated as

External work done due to unit load = $(1/2) \times 1 \times \delta$

Internal strain energy stored due to unit load = $(1/2)\delta_u dA du$

We have

$$(1/2) \times 1 \times \delta = (1/2)\Sigma\sigma_u dA du \qquad (10.2)$$

If we now apply the loads P_i gradually, the deformation under M will be Δ and the deformation under P_i will be Δ_i.

External work done = $1\Delta + (1/2)\Sigma P_i\Delta_i$

You will note that both linear relationship between loads and deformations and the law of superposition are applied here.

Total strain energy stored in the beam

$$= (1/2)\Sigma\sigma_u dA du + \Sigma\sigma_u dA\,dL + (1/2)\Sigma\sigma dA\,dL$$

By law of conservation of energy,

$$(1/2)1\ \delta + (1/2)\Sigma P_i\Delta_i + 1\ \Delta = (1/2)\sigma_u dA du + (1/2)\Sigma(\sigma dA\,dL) + (1/2)\Sigma\sigma_u dA\,dL \qquad (10.3)$$

Subtracting Equations (10.1) and (10.2) from Eq. (10.3), we get

$$1\Delta = \Sigma\sigma_u dA\,dL = \Sigma F_u dL$$

where F_u is the force on the element of area dA.

The force in the element can be due to bending moment, axial force, etc., depending upon the structure being analysed. The deformation sought can be deflection, slope, etc. In the case of angular deformations, we apply a unit moment in place of a unit point load.

Limiting the discussion to deformation due to bending alone, the term ΣFdL is the strain energy stored in the beam due to bending moment. This, as we have derived earlier, is given as $b\int M_x^2 dx/EI$.

We apply this to a beam carrying loads as shown in Fig. 10.50. We have the basic equation of unit load method, which gives $1\Delta = \Sigma F_u dL$. In this equation, Ä is the deflection at the point where the unit load is applied. Also F_u is the force in the element of area dA due to the stress σ_u caused by the unit load and dL is the deformation of the element of length dx due to the applied loads P_i.

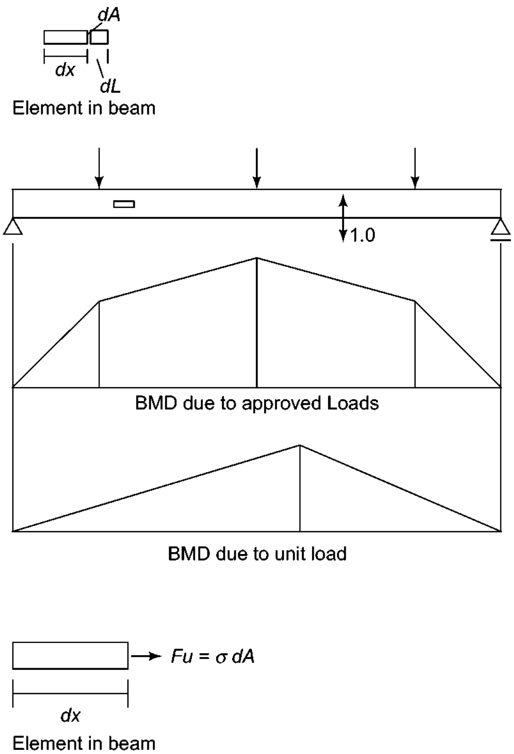


Fig. 10.50 Unit load method for a beam

If m is the moment caused by the unit load at the section of the element, then

$$F_u = (my/I)dA$$

The terms are explained in the figure.

If M is the moment caused by the loads P_i at the same section, then

$$dL = (My/I)dx/E$$

We can now write

$$1\Delta = \Sigma F_u dL = \int_0^L \int_0^A (mydA/I)(Mydx/EI)$$

$$= \int_0^L (Mmdx/EI^2) \int_0^A y^2 dA$$

$$= \int_0^L \frac{Mmdx}{EI} \text{ as } \int_0^A y^2 dA = I$$

This is the basic equation for the unit load method. The application of this equation to beam deflections and slopes is illustrated through the following examples. The application to truss deflections is given in Chapter 13.

Example 10.32 Unit load method: cantilever with uniformly varying load

A cantilever of 3 m span carries a uniformly varying load varying from 18 kN/m at the fixed end to zero at the free end. Find the deflection and slope at the free end using the unit load method. EI = 40,000 kNm^2.

Solution The beam with the load is shown in Fig. 10.51.

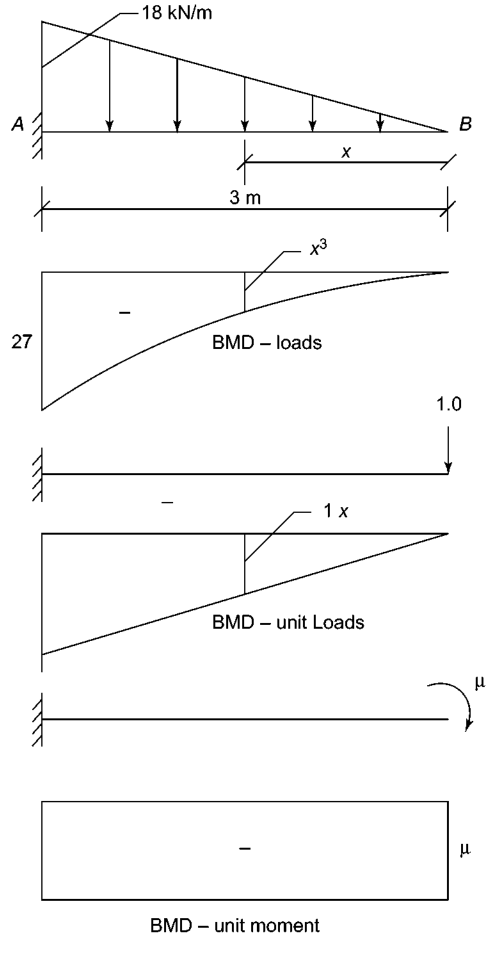


Fig. 10.51

$$m_x = -1x$$

$$y_B = \int_0^3 (-x^3)(-x)dx/EI = [x^5/5EI]_0^3 = \frac{3^5}{5\times 40{,}000}\text{ m} = \frac{243\times 1000}{5\times 40,000} = 1.2\text{ mm}$$

To find the slope at B, we apply a unit couple of 1 kNm at B.

Here m_x = 1 kNm is a constant.

$$\text{Slope at } B,\ \theta_B = \int_0^3 (-x^3)dx/EI = \frac{[-x^4]_0^3}{(4E)} = \frac{81}{4(40,000)} = 0.506 \times 10^{-3} \text{ radians}$$

Example 10.33 Unit load method: overhanging beam

The overhanging beam carries the loads as shown in Fig. 10.52. Find the deflection under the loads and the slope at A.

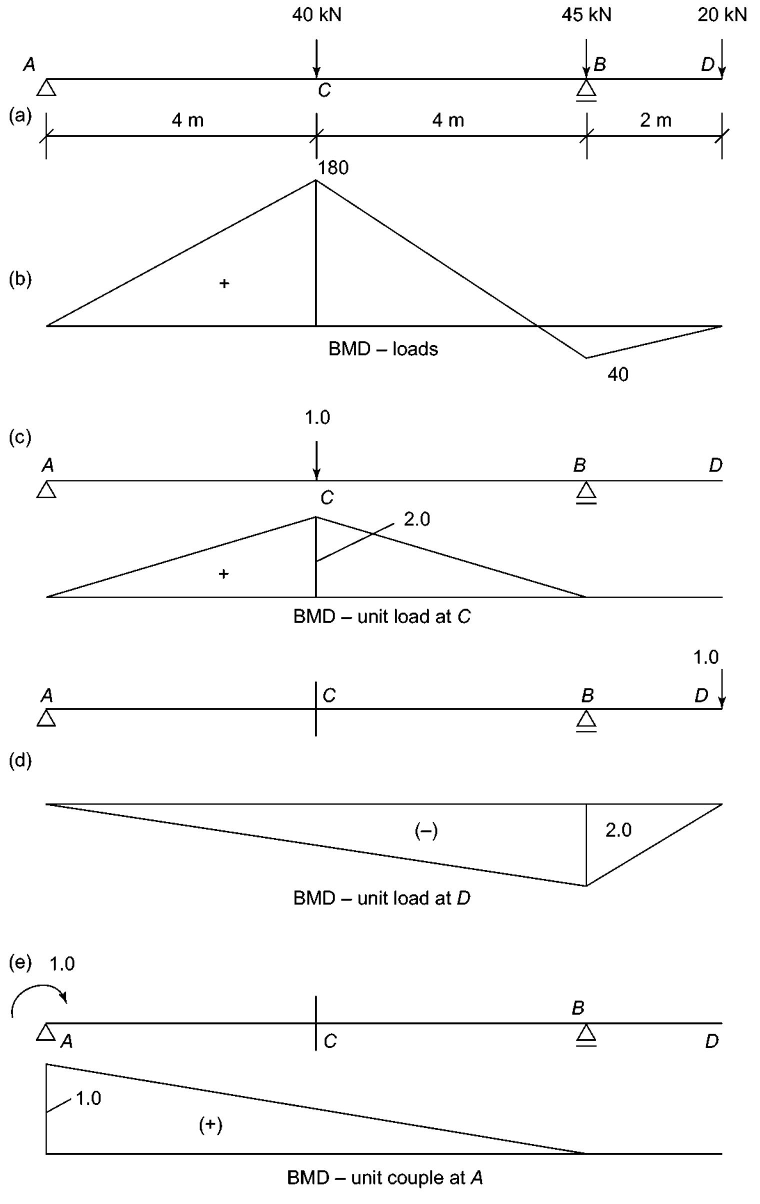


Fig. 10.52 (*contd*)

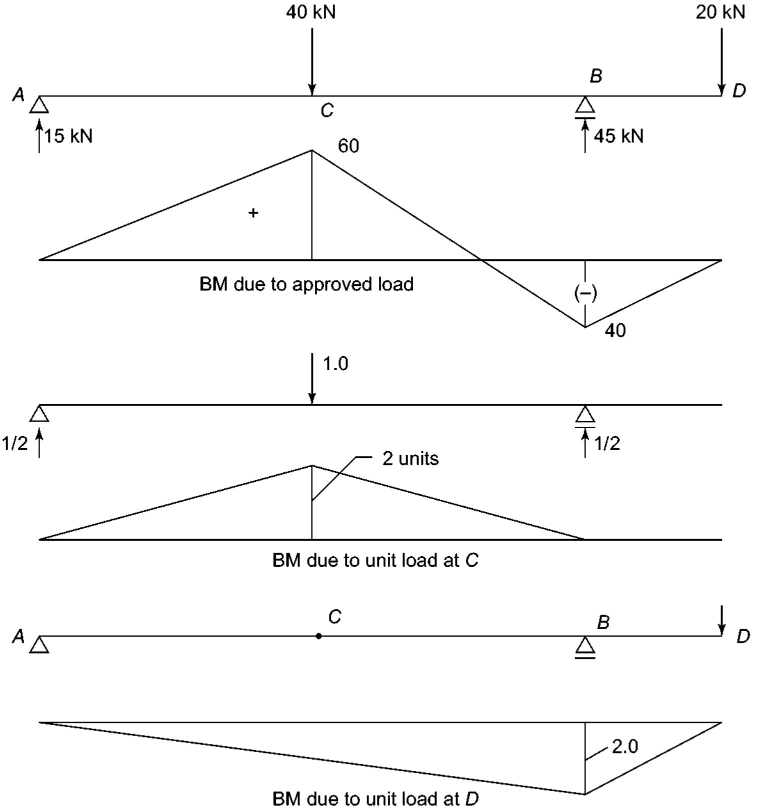


Fig. 10.52

Solution The bending moment diagram due to the applied loads is shown in Fig. 10.52(b). To find the deflection at C, we apply a unit load at C. The bending moment diagram due to the unit load is shown in Fig. 10.52(c). We have to consider the integration of the expression ($Mmdx/EI$) in three parts. Table 10.4 shows the moments.

Table 10.4

Segment	M_x	*M*	*Origin*	*Range*
AC	$15x$	$x/2$	*A*	0 to 4
CB	$15x - 40(x-4)$	$(1/2)x - (x-4)$	*B*	4 to 8
BD	$-20x$	0	*D*	0 to 2

As $m = 0$ in the segment DC, integration need not be done for that segment.

Vertical deflection at C, $y_C = \int_0^4 15x(x/2)dx/EI + \int_0^8 (15x - 40(x-4))[(1/2)x - (x-4)]dx/EI$

$$y_C = \int_0^4 \frac{15x^2 dx}{EI} + \int_4^8 \frac{[(160 - 25x)(40 - x/2)dx}{EI}$$

$$= [15x^3/6EI]_0^4 + \frac{[640x - 90x^2 + 25x^3/6]_4^8}{EI}$$

$$= 266.67/EI \downarrow$$

To find the deflection under the 20 kN load, we apply a unit load at D. The bending moments at different segments are as shown in Table 10.5.

Table 10.5

Segment	M_x	*M*	*Origin*	*Range*
AC	$15x$	$-x/4$	*A*	0 to 4
CB	$15x - 40(x - 4)$	$-x/4$	*C*	4 to 8
BD	$-20x$	$-x$	*D*	0 to 2

$$y_D = \int_0^4 [15x(-x/4)dx/EI + \int_0^8 [(15x - 40(x-4))(-x/4)]dx/EI$$

$$+ \int_0^2 [(-20x(-x))dx/EI]$$

$$= \frac{[-15x^3/12]_0^4}{EI} + \frac{[-20x^2 + 25x^3/12]_4^8}{EI} + \frac{[20x^3/3]_0^2}{EI}$$

$$= -53.33/EI \uparrow$$

To find the slope at *A*, we apply a clockwise couple as shown. The bending moment in the different segments will be as shown in Table 10.6. As $m = 0$ in the segment *BD*, it is not required to be integrated.

Table 10.6

Segment	M_x	*M*	*Origin*	*Range*
AC	$15x$	$1 - x/8$	*A*	0 to 4
CB	$15x - 40(x - 4)$	$1 - x/8$	*C*	4 to 8

$$EI\theta_A = \int_0^4 [15x(1 - x/8)dx + \int_0^8 [(15x - 40(x-4))\,(1 - x/8)]dx$$

$$= [15x^2/2 - 15x^3/24]_0^4 + [160x - 45x^2/2 + 25x^3/24]_0^8$$

$$= 120 - 40 + [1280 - 1440 + 533.33 - 640 + 360 - 66.67]$$

$$= 106.67$$

$$\theta_A = 106.67/EI \text{ (clockwise as assumed)}$$

Example 10.34 Unit load method: cantilever frame

In the cantilever frame loaded as shown in Fig. 10.53, find the horizontal and vertical deflection at *D*.

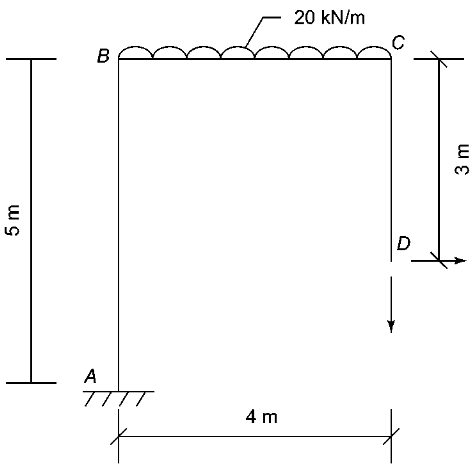


Fig. 10.53

Solution To find the vertical deflection of point D, apply a vertically downward unit load at D. The bending moments in the different segments are as shown in Table 10.7.

Table 10.7

Segment	M_x	M	*Origin*	*Limits of integration*
DC	0	0	D	0 to 3
CB	$-20x^2/2$	$-x$	C	0 to 4
BA	-160	-4	B	0 to 5

Deflection at D, $y_D = \Sigma \int M_x m dx/EI$

$$y_D = \int_0^4 \frac{(-10x^2)(-x)dx}{EI} + \int_0^5 \frac{(-160x^2)(-4)dx}{EI} = \frac{[10x^4]_0^4}{EI} + \frac{[640x]_0^5}{EI}$$

$$= 3840/EI$$

To find the horizontal deflection at D, apply a horizontal unit load as shown in the figure. The moments are as shown in Table 10.8.

Table 10.8

Segment	M_x	M	*Origin*	*Range*
DC	0	$1.x$	D	0 to 3 m
CB	$-10x^2$	$+3$	C	0 to 4 m
BA	-160	$(3 - 1.x)$	B	0 to 5 m

Horizontal deflection, $y_H = \Sigma \int Mmdx/EI$

$$EI\, y_H = \int_0^4 -30x^2 dx + \int_0^5 (-480 + 160x)dx = [-10x^3]_0^4 + [-480x + 80x^2]_0^5$$

$$= -1040$$

Horizontal deflection, $y_H = -1040/EI$ (The negative sign shows that the deflection is in a direction opposite to that of the unit load.) ■

Example 10.35 Unit load method: cantilever

The cantilever beam shown in Fig. 10.54 carries a UD load of 40 kN/m over the whole span. Find the deflection and slope at the free end.

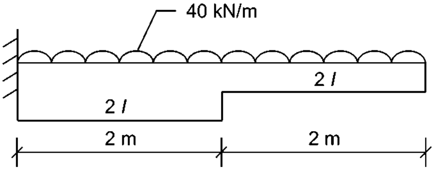


Fig. 10.54

Solution Due to change in value of MI, integration has to be done in two parts. The moments are shown in Table 10.9.

Table 10.9

Segment	M_x	*m*	*MI*	*Origin*	*Range*
BC	$-20x^2$	$-1.x$	*I*	*B*	0 to 2
CA	$-20x^2$	$-1.x$	2*I*	*C*	2 to 4

Vertical deflection at B, $y_B = \Sigma \int M_x m dx/EI$

$$y_B = \int_0^2 \frac{(-20x^2)(-x)dx}{EI} + \int_2^4 \frac{(-20x^2)(-x)dx}{EI} = \frac{[20x^4]_0^2}{EI} + \frac{[20x^4]_0^4}{EI}$$

$$= 680$$

Vertical deflection of B = 680/*EI*

To find the slope at B, apply a unit clockwise couple as shown at B. The bending moments will be as shown in Table 10.10.

Table 10.10

Segment	M_x	*m*	*MI*	*Origin*	*Range*
BC	$-20x^2$	-1	*I*	*B*	0 to 2
CA	$-20x^2$	-1	2*I*	*C*	2 to 4

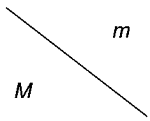
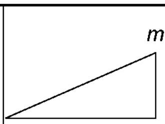
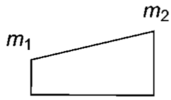
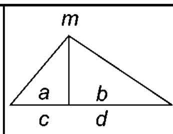
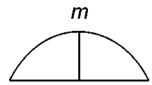
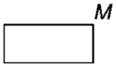
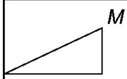
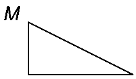
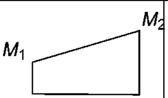
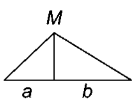
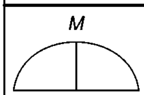
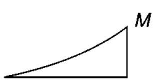
Slope at B, $\theta_B = \Sigma \int M_x m dx/EI]$

$$EI\,\theta_B = \int_0^2 20x^2 dx + \int_0^4 (20x^2/2)dx = [20x^3/3]_0^2 + [20x^3/6]_2^4 = 240$$

$$\theta_B = 240/EI$$

Table of product integrals In the unit load method, we find the integral $\int Mmdx/EI$. In most of the structures, material property like E and geometric property I remain constant over the length of the member. EI can then be taken out of the integral sign. We then have $(1/EI)\int Mmdx$. In this integral, M is moment at x due to the applied loads and m is the moment at the same section due to the unit load or couple. To facilitate the computation of such integrals, the integrals shown in Table 10.11 can be used. Note that the integrals are only for straight line and quadratic equations. The figures shown are bending moment diagrams. We illustrate the use of the table with two examples.

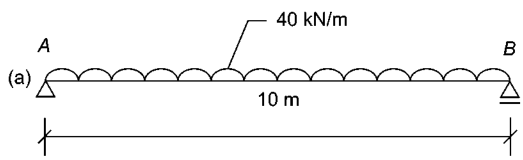
Table 10.11 Table of integrals of product *Mm*

				
	$\frac{L}{2}Mm$	$\frac{L}{2}(m_1+m_2)M$	$\frac{L}{2}(mM)$	$\frac{2L}{3}mM$
	$\frac{L}{3}Mm$	$\frac{L}{6}(m_1+2m_2)M$	$\frac{L}{6}\left(1+\frac{a}{L}\right)mM$	$\frac{L}{3}mM$
	$\frac{L}{6}mM$	$\frac{L}{6}(2m_1+m_2)M$	$\frac{L}{6}\left(1+\frac{b}{L}\right)mM$	$\frac{L}{3}mM$
	$\frac{L}{6}m(m_1+2m_2)$	$\frac{L}{6}m_1(2m_1+m_2)$	$\frac{L}{6}\left(1+\frac{b}{L}\right)\mathrm{mM}_1$	$\frac{L}{6}m(M_1+M_2)$
		$+\frac{L}{6}m_2(M_1+2M_2)$	$+\frac{L}{6}\left(1+\frac{a}{L}\right)mM_2$	
	$\frac{L}{6}\left(1+\frac{a}{L}\right)mM$	$\frac{L}{6}\left(1+\frac{b}{L}\right)m_1M$ $+\frac{L}{6}\left(1+\frac{a}{L}\right)m_2M$	For $< \leq a$ $\frac{L}{3}mM -$ $\frac{L(a-c)^2}{6ad}\mathrm{mM}$	$\frac{L}{3}\left(1+\frac{ab}{L^2}\right)mM$
	$\frac{L}{3}mM$	$va\frac{L}{3}(m_1+m_2)M$	$\frac{L}{3}\left(1+\frac{ab}{L^2}\right)mM$	$\frac{8L}{15}mM$
	$\frac{1}{4}mM$	$\frac{L}{12}(m_1+3m_2)M$	$\frac{L}{12}\left(1+\frac{a}{L}+\frac{a^2}{L^2}\right)$ $\times(m.\ M)$	$\frac{L}{5}Mm$

Example 10.36 Unit load method: table of integrals

The simply supported beam of span 10 m carries a UD load 40 kN/m over the whole span. Find the deflection at the centre of the span and the slope at the left support. *EI* is constant and is equal to 10^6 kNm^2.

Solution The beam with the loading and the BM diagram are shown in Figures 10.55(a) and (b).



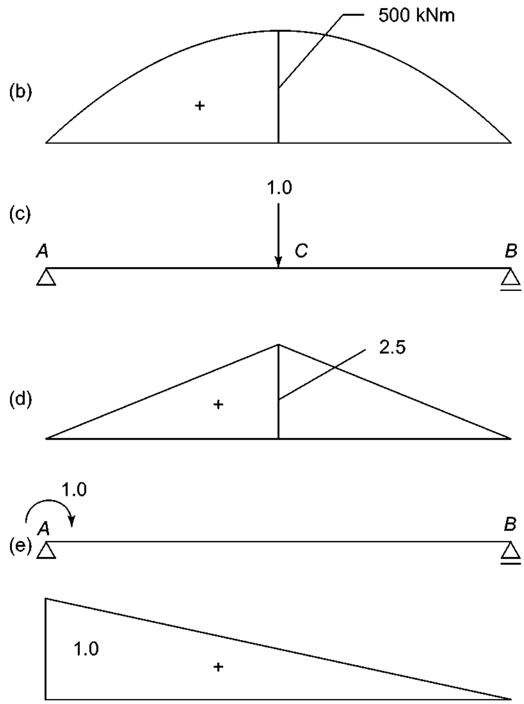


Fig. 10.55

To find the deflection at the centre of the span, we apply a unit load at C, the centre of the beam. The BM diagram due to the unit load is shown in Fig. 10.55(d). Now,

$$\text{Deflection at the centre} = \int (Mmdx/EI) = (1/EI)\int Mmdx$$

This integral can be found from Table 10.11. For the bending moment diagram in the form of a parabola and the unit load BM diagram in the form of a triangle, we have

$$\int Mmdx = L(1 + ab/L^2)m_u M_L/3$$

From the bending moment diagrams, $m_u = L/4$ and $M_L = wL^2/8$; $a = b = L/2$. Therefore,

$$\int Mmdx = L(1 + L^2/4L^2)(L/4)(wL^2/8) = 5/384wL^4$$

$$\text{Deflection at the centre} = \frac{5wL^4}{384EI} = \frac{5 \times 40 \times 10^4}{384 \times 10^6} = 5.2 \text{ mm}$$

To find the slope at A, we apply a unit couple at A as shown. The bending moment diagram due to this couple is shown in Fig. 10.55(e). Now,

$$\text{Slope } \theta_A = (1/EI)\int Mmdx$$

From Table 10.11,

$$\int Mmdx = (L/3)M_L m_u = (L/3)(wL^2/8)(1) = wL^3/24$$

Slope at A, $\theta_A = \dfrac{40(10)^3}{24 \times 10^6} = 0.002$ radians

Example 10.37 Unit load method: table of integrals

A simply supported beam, of 8 m span, carries two point loads of 30 kN and 40 kN at 2 m and 4 m from the left end. Find the deflection under the loads and the slope at B. $EI = 0.4 \times 10^6$ kNm^2.

Solution The beam with the loads is shown in Fig. 10.56(a).

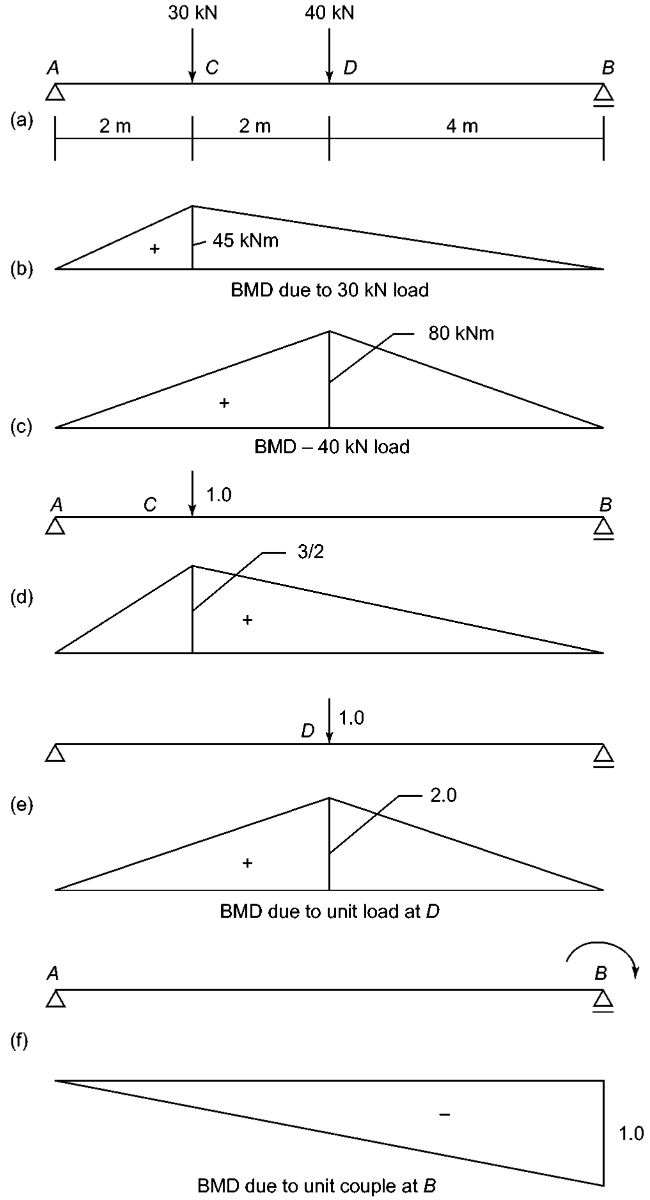


Fig. 10.56

We consider the two point loads separately. The bending moment diagrams due to the 30 kN load are shown in Figures 10.56(b) and (c).

(i) *Deflection under 30 kN load* To find the deflection under the 30 kN load, we apply a unit load at *C*. The BM diagram due to the unit load is shown in figure 10.56 (d). The integral of the product of this diagram with the two load diagrams is given by

$$EI\, y_C = \int Mmdx = [(L/3)(M_{30}m) + (L/3)M_{40}m - L(a-c)^2 M_{40}m/(6ad)$$

$$= \frac{8\times 45\times 1.5}{3} + \frac{8\times 80\times 1.5}{3} - \frac{8(4-2)^2\times 80\times 1.5}{6\times 4\times 6} = 473.33$$

Deflection at $C = 473.33/EI = 473.33/(0.9\times 10^6) = 1.2$ mm

(ii) *Deflection under 40 kN load* We apply a unit load at point *D*. The BM diagram due to this load is shown in Fig. 10.56(e). The deflection under the 40 kN load is

$$EI\, y_D = \int Mmdx = (L/3)M_{30}m - L(a-c)^2 M_{30}m/(6ad) + (L/3)M_{40}m$$

$$= \frac{8\times 45\times 2}{3} - \frac{8\,(4-2)^2\times 45\times 2}{6\times 4\times 6} + \frac{8\times 80\times 2}{3}$$

$$= 240 - 15 + 426.67 = 646.67$$

$$y_D = 646.67/0.4\times 10^6 = 1.6 \text{ mm}$$

(iii) *Slope at B* We apply a clockwise couple at *B* as shown in Fig. 10.56(f). The unit load *m*-diagram is a triangle with the value of –1.0 at *B*. The slope at *B* is θ_B and is given by

$$EI\,\theta_B = (L/6)(1 + a/L)M_{30}m + (L/6)(1 + a/L)M_{40}m$$

$$= (8/6)(1 + 2/8)45\times 1 + (8/6)(1 + 4/8)80\times 1 = 235$$

Slope at B, $\theta_B = 235/(0.4\times 10^6) = 0.6\times 10^{-3}$ radians

10.9.4 Maxwell's Reciprocal Theorem

Maxwell's reciprocal theorem, also called Maxwell–Betty theorem, states the relationship between the deformations between two points in a body due to load systems acting at the two points. As an example, referring to Fig. 10.57, a simple way to state the theorem is as follows:

The deflection at point A due to load P acting at point B is equal to the deflection at point B due to the same load P acting at point A.

The theorem applies to slopes as well as to slopes and deflections. We first prove the simple case. Consider the case shown in Fig. 10.57. The deflection at point *A* when load *P* is acting at *B* is Δ_{AB}. The first subscript refers to the point where the deflection is measured and the second subscript to the point where the load is placed. Thus, Δ_{BA} is the deflection of point *B* when load *P* is acting at point *A*.

When the load is at A, work done by the load = $(1/2)P\Delta_{AA}$

When the load is at B, work done by the load = $(1/2)P\Delta_{BB}$

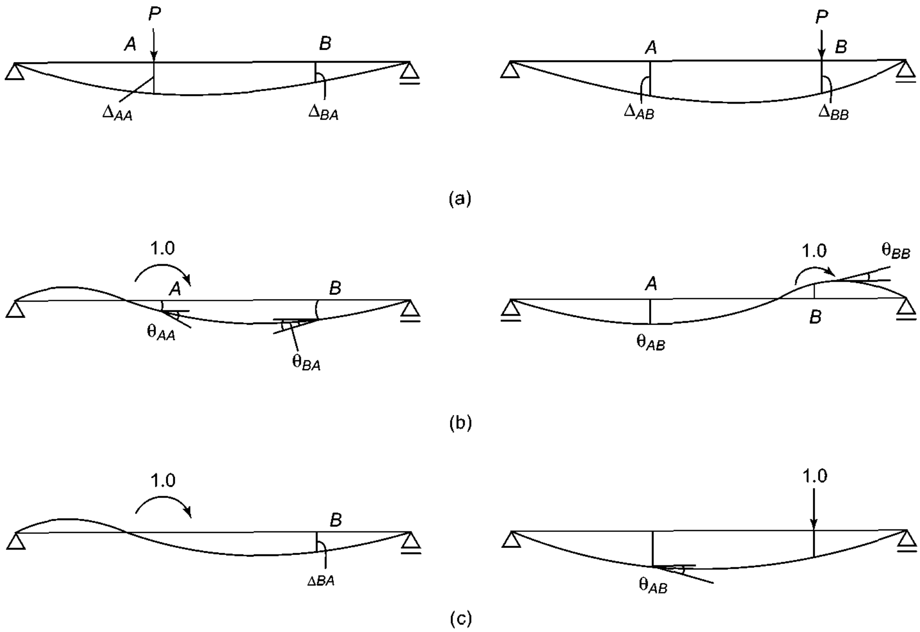


Fig. 10.57

Let load P be first gradually applied at A and then an equal load is gradually applied at B. Then the work done can be expressed as

$$W_1 = (1/2)P\Delta_{AA} + (1/2)P\Delta_{BB} + P\Delta_{AB}$$

as the load is already acting at A when the load is applied at B

Now, consider the reverse case of load P being applied first at B and then an equal load being applied at A. The work done in this case can be expressed as

$$W_2 = (1/2)P\Delta_{BB} + (1/2)P\Delta_{AA} + P\Delta_{BA}$$

As the work done must be equal irrespective of the sequence of the applying loads, $W_1 = W_2$.

$$(1/2)P\Delta_{AA} + (1/2)P\Delta_{BB} + P\Delta_{AB} = (1/2)P\Delta_{BB} + (1/2)P\Delta_{AA} + P\Delta_{BA.}$$

We get $\Delta_{AB} = \Delta_{BA}$

Now consider the case of couple M being applied in a similar fashion as shown in Fig. 10.57(b). If the rotational displacement at A when the couple is applied at B is θ_{AB} and the rotational displacement at B when the couple is placed at A is θ_{BA}, we can prove on similar lines that $\theta_{AB} = \theta_{BA}$.

Now consider the third case shown in Fig. 10.57(c). Here a unit couple is applied at point A. The rotation at A is θ_{AA} and the deflection at B is Δ_{BA}. We now apply a unit load at B. The deflection at B is Δ_{BB} and the rotation at A is θ_{AB}. The work done in this process is $W_1 = (1/2)\theta_{AA} + (1/2)\Delta_{BB} + \theta_{AB}$.

Now consider the reverse process of applying the unit load at B first and then applying the unit couple at A. Using the same notations, work done in this case is

$$W_2 = (1/2)\Delta_{BB} + (1/2)\theta_{AA} + \Delta_{BA}$$

As the total work done by these two processes must be the same, $W_1 = W_2$. This gives

$$\theta_{AB} = \Delta_{BA}$$

This can be stated as follows:

The rotational displacement at point A due to a load P at B is equal to the linear displacement at B due to an equivalent couple at A.

We have proved this with unit load and unit couple. The result holds good if the load and the couple are equal numerically.

The reciprocal theorem can be generalized as follows:

In a stable structure with linear load deformation relationship, the virtual work done by a system of loads/couples, constituting the first system, in going through the displacements caused by the loads/couples of a second system is equal to the virtual work done by the loads/couples of the second system in going through the corresponding deformations caused by the first system.

10.9.5 Betty's Law

The above general form of the reciprocal theorem is known as Betty's law. It is a generalization of Maxwell's reciprocal theorem. Considering the beam subjected to two systems of forces and couples as shown in Fig. 10.58, forces P_1, P_2, P_3 and couples M_1, M_2, M_3 form the first system F_1. Forces W_1, W_2, W_3 and couples μ_1, μ_2, μ_3 form the second system F_2.

Let W_1 be the work done by the first system of forces when acting alone.

Let W_2 be the work done by the second system of forces acting alone.

Let W_{12} be the work done by the first system of forces when they move through displacements and rotations caused by the second system of forces at their points of application.

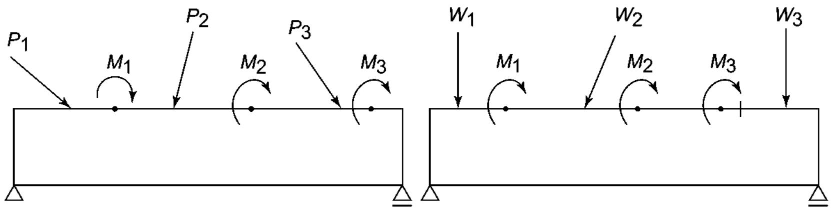


Fig. 10.58 (a)

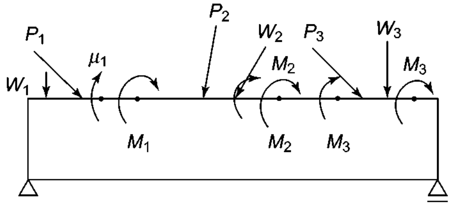


Fig. 10.58 (b)

Let W_{21} be the work done by the second system of forces when they move through displacements and rotations caused by the first system of forces at their points of application.

Refer to Fig. 10.58(b). We apply the system of forces F_1 alone first and then apply the second system of forces F_2.

Total work done in this case = $W_1 + W_2 + W_{12}$

If we apply the system of forces F_2 first and then the system of forces F_1, then

Total work done = $W_2 + W_1 + W_{21}$

As the total work done should remain the same irrespective of the order of applying the forces, we can equate the two and get $W_{12} = W_{21}$, which is the statement of Betty's law.

Summary

When a structural element is subjected to a straining action, the point of application of the external load moves due to the deformation of the structure and does work. This work is stored in the element as strain energy. Strain energy is, thus, the negative work done by the internal stress resultants, and has a direction opposite to that of deformation. It is equal to the area under the load-deformation diagram for the element.

Under an axial load, when the load is applied gradually,

$$\text{Strain energy } U = \frac{P^2L}{2AL} = \frac{\sigma^2 AL}{2E} = \frac{\sigma^2}{2E} \times \text{volume}$$

The maximum strain energy stored in a material per unit volume without permanent set (i.e., when the stress is within the elastic limit) is known as modulus of resilience. When the axial load is suddenly applied, the stress and deformation are double the values when the same load is gradually applied; the strain energy is consequently four times that in the gradually applied case. Vibrations are set up in the member due to this suddenly applied load.

When the axial load is applied with an impact, the kinetic energy of the body and the external work done due to deformation are converted into instantaneous strain energy. Vibrations are set up in the member due to large instantaneous deformation. The stresses and deformations can be calculated by equating strain energy to the energy of the striking body and the external work done.

Toughness is a characteristic of the material of its ability to resist impact loading and depends on the ductility of the material. Mild steel has more toughness than high-strength steel because it is ductile and can undergo large deformation and absorb large amounts of energy.

In the case of a member subjected to bending, the strain energy is given by $U = \int M^2 dx/2EI$, where the integral is over the whole length of the beam. The strain energy due to shear is given by $U = \tau^2/2G = r^2G/2$ for a unit volume, considering the SF to be constant. For varying SF, as in the case of a beam subjected to lateral loads, the strain energy stored is calculated by integration, taking into account the variation of shear stress over the cross section and the SF over the length. The strain energy due to torsion in the case of a circular shaft of length l subjected to torque T is given by $U = T^2 l/2GJ = GJ\theta^2/2l$. Strain energy concepts are generally employed to calculate deformations. The methods are particularly useful in the case of members subjected to impact loading.

The theorems relating to strain energy are the theorem of conservation of energy, Castigliano's theorem, and Engesser's theorem. Castigliano's theorem and unit load method are useful in determining deflections and slopes in beams and frames.

Exercises

Review Questions

1. Define the terms strain energy, modulus of resilience, and toughness.
2. State from everyday experience a system wherein strain energy is stored in an element and is recovered for convenient use.
3. For a prismatic bar subjected to an axial load, state the expression for strain energy, in terms of (i) load, (ii) stress, and (iii) elongation of the bar.
4. How is the strain energy related to the load-deformation diagram of a bar?
5. Within the proportional limit, does the strain energy of bar vary linearly with the load? Explain.
6. Mild steel has more toughness than high-strength steel. Explain in terms of strain energy.
7. If the load on an axially loaded bar is doubled, by how much does the strain energy increase?
8. Two bars are made of the same material and have the same cross-sectional area. Find the ratio of their lengths, if their strain energies are equal when one is subjected to a gradually applied load and the other to the same load suddenly applied.
9. If two bars have the same sectional area and length, and are subjected to the same load, find the ratio of the strain energies stored in them, if E for one is two and a half times that of the other.
10. If two bars have the same mass, will the strain energies be the same under the same load?
11. One bar is half as long as the other, and they are made of the same material. Under the same load, if their strain energies are also in the ratio 1 : 2, what is the ratio of their cross-sectional areas.
12. Give the expressions for strain energy due to axial load, BM, shear, and torsion. Discuss the similarity in the expressions for strain energy in all cases.
13. Explain the behaviour of a bar subjected to (i) a suddenly applied load, and (ii) an impact load.
14. State the advantages of calculating the strain energy of a bar. What is its application in structural analysis?
15. State the theorem of conservation of energy.
16. State Engesser's theorem of complimentary energy. State the advantages of this theorem.
17. State and prove Castigliano's first theorem.
18. State and prove the unit load method for finding deflections.
19. State and prove Maxwell–Betty reciprocal theorem.

Problems

1. Calculate the strain energy stored in a bar 2 m long, 50 mm wide, and 40 mm thick, when it is subjected to a tensile load of 50 kN. Take $E = 200$ GPa.
2. A rectangular body 500 mm long, 100 mm wide, and 50 mm thick subjected to a shear stress of 80 MPa. Determine the strain energy stored in the body. $G = 85$ GPa.

3. Find the strain energy of a steel rod, of diameter 20 mm and length 400 mm, when it is subjected to an axial load of 40 kN. What is the elongation of the rod under this load? Determine the modulus of resilience of the rod if the elastic limit is 250 MPa. Take E = 200 GPa.
4. A stepped steel rod is in two parts—one has a diameter of 20 mm and a length of 1 m, and the second a length of 800 mm and a diameter of 12 mm. Find the strain energy of the bar under a load of 15 kN. Take E = 200 GPa.
5. Find the strain energy of an aluminium bar which is in three sections—one of 20 mm ϕ and length 1.2 m, another of diameter 15 mm and length 0.8 m, and the last of 12 mm ϕ and length 0.6 m. Take E = 70 GPa. The load is such that the maximum stress in the bar is 35 MPa.
6. A solid rod tapers from a diameter D to d over a length L. If it is subjected to an axial load P, show that its strain energy is

$$\frac{P^2 L}{2Ea}\frac{d}{D}$$

where a is the area of the smaller end of diameter d and E is Young's modulus of elasticity of the material.
7. A solid rod of diameter 20 mm and length 200 mm is subjected to an axial load of 20 kN, which is suddenly applied to the rod. Find the maximum stress and maximum instantaneous elongation. E = 200 GPa.
8. A plate of uniform thickness 5 mm has a tapering width varying from 50 mm to 25 mm over a length of 250 mm. It is subjected to an axial pull of 30 kN. Find the strain energy stored in the bar and the extension in length. Take E = 200 GPa.
9. A steel rod of solid circular section and diameter 20 mm is covered by a brass sleeve of internal diameter 20 mm and thickness 5 mm. The composite rod of length 200 mm is subjected to axial load such that the stress in the brass is not more than 50 MPa. Find the elongation of the bar and the strain energy stored under this load. Take E = 200 GPa for steel and 100 GPa for brass.
10. A bar 400 mm long and of diameter 500 mm is subjected to a tensile load of 200 kN. Find the stress, elongation, and strain energy produced if the load is applied gradually. What would be the instantaneous stress and elongation if the same load is applied suddenly. Take E = 200 GPa.
11. In the plane frameworks shown in Fig. 10.59, find the strain energy stored in the systems. The bars are of uniform section and have the cross-sectional areas indicated. Take E = 200 GPa.

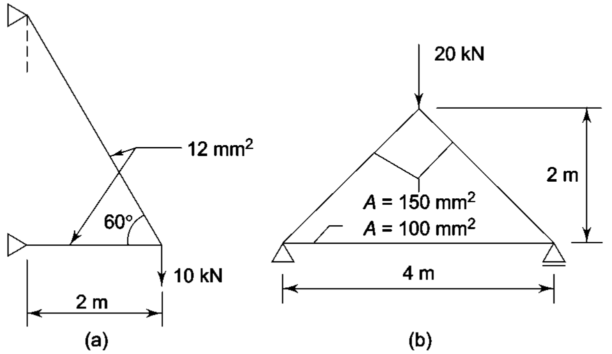


Fig. 10.59

12. The brass rod shown in Fig. 10.60 has a diameter of 20 mm. The collar *D* moves along the rod and strikes the metal plate attached at the lower end. If E = 100 GPa, determine the strain energy and instantaneous elongation when (i) the collar *D* weighs 1000 N and falls through a height of 5 mm, and (ii) the collar weighs 10 N and falls through a height of 500 mm.

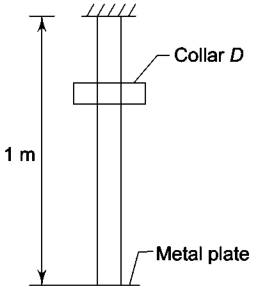


Fig. 10.60

13. A bar of length 400 mm and diameter 50 mm is subjected to a tensile load of 20 kN. Find the stress, elongation, and strain energy produced if the load is applied gradually. What would be the instantaneous stress and elongation if the same load is applied suddenly. Take E = 200 GPa.
14. In the system shown in Fig. 10.61, the collar falls through a height of 1 m. Find the maximum weight of the collar such that the stress in the material does not exceed 200 N/mm^2. Take E = 200 GPa.

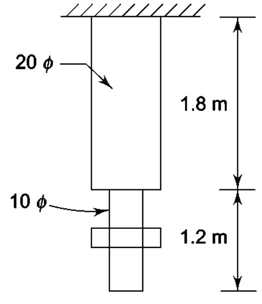


Fig. 10.61

15. A bar is made up of a brass rod and a steel rod joined end to end as shown in Fig. 10.62. If the collar weighs 50 N, find the maximum height of fall so that the permissible stress in either steel or brass is not exceeded. Permissible stresses are 50 MPa for brass and 150 MPa for steel. Take E = 100 GPa for brass and 200 GPa for steel.

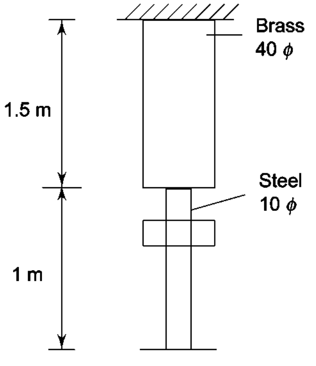


Fig. 10.62

16. In the system shown in Fig. 10.63, find the mass of the body that strikes the rod axially with a velocity of 2 cm/s. The rod is of brass with E = 100 GPa, and the maximum stress should not exceed 50 MPa.

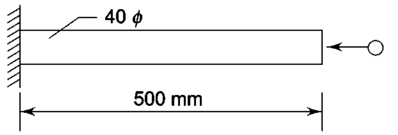


Fig. 10.63

17. A vertical tie rod rigidly fixed at the top end consists of a steel rod of 2 m length and 20 mm diameter encased throughout in a brass tube of 20 mm internal diameter and 30 mm external diameter. The rod and the casing are fixed together at the ends. The compound bar is suddenly loaded in tension by a mass of 1530 kg falling freely through 3 mm before being arrested by the tie. Compute the maximum stresses in steel and brass. Take $E_s = 2 \times 10^5$ MPa and $E_b = 1 \times 10^5$ MPa.
18. A bar of length 3 m has an area of cross section of 2400 mm^2 for a length of 1.8 m and an area of cross section of 1200 m^2 for the remaining length. Find the strain energy stored in the bar when a load of 80 kN is gradually applied. Determine also the ratio of the strain energy stored in a bar of uniform section and of the same length and volume under the same load. Take E = 200 GPa.
19. A composite bar is made up of a steel rod of diameter 20 mm, and a brass tube of external diameter 30 mm and thickness 5 mm. The bar is 1 m long and is subjected to impact due to a load of 50 N falling from a height of 200 mm. Calculate the instantaneous stresses in each material. Take E = 200 GPa for steel and 100 GPa for brass.
20. A cantilever beam of span 2 m carries a load of 1 kN at the free end. If its cross section is 80 mm wide and 120 mm deep, find the strain energy stored in the beam due to bending. What is the deflection at the end of the cantilever under this load? Take E = 200 GPa.
21. A steel pipe of external diameter 30 mm and thickness 5 mm is supported at its ends which are 2 m apart. Find the strain energy due to bending, stored in the pipe. E = 200 GPa. The pipe carries a load of 20 kN/m of its length in addition to its own weight. Assume that the weight is 78.5 kN per m^3.
22. A beam of span 3 m and section 30 mm × 60 mm carries a load varying uniformly from zero at one end to 30 kN/m at the other. Find the ratio of the strain energies stored in the beam due to bending and shear. Take E = 200 GPa and G = 80 GPa.
23. A cantilever beam has a span of 2 m and is of rectangular section—of breadth 30 mm and depth 60 mm. A load of 50 N falls at the free end from a height h. Find the maximum value of h so that the maximum stress in the beam does not exceed 200 MPa. What is the maximum instantaneous deflection in such a case. Take E = 200 GPa.
24. A beam supported over a span of 6 m has a maximum deflection of 2 mm when a load of 5 kN is applied at its mid-point. If the same load is dropped from a height of 20 mm at the mid-point of the beam, what will be the maximum deflection?
25. A simply supported beam has a span of 3 m. A concentrated load of 30 kN is applied at mid-span gradually. If E = 200 GPa and G = 80 GPa and the beam is of rectangular section (60 mm × 150 mm), find the strain energies stored in the beam (i) due to bending and (ii) due to shear. Also find the deflection under the load.

26. In the arrangement shown in Fig. 10.64, find the vertical deflection of the loaded point P. Take $E = 200$ GPa.

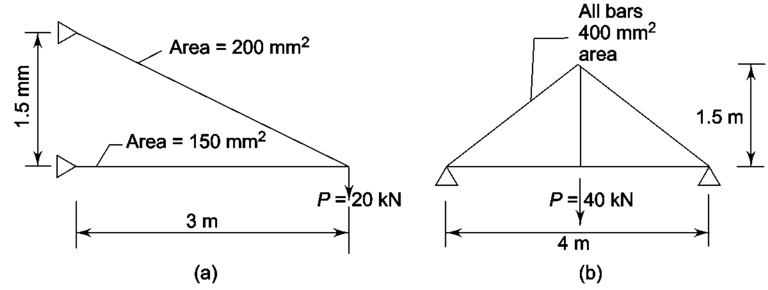


Fig. 10.64

27. A cantilever of length L carries a UD load of w/m for half its length from the fixed end. Determine the deflection at the free end. EI is constant.
28. The two bars AB and AC support a load of 20 kN as shown in Fig. 10.65. Find the vertical deflection of point A under the load.

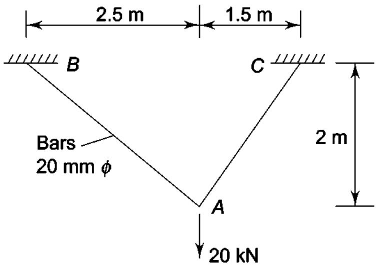


Fig. 10.65

29. In the frame shown in Fig. 10.66, find the horizontal deflection of point C under the load of 30 kN. The bars have an area of 600 mm^2. $E = 200$ GPa.

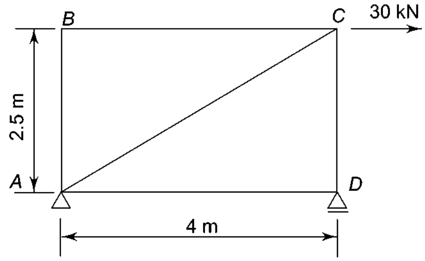


Fig. 10.66

30. A simply supported beam has a span of 6 m. It is subjected to a clockwise couple load of 48 kN m at 2 m from the left end. If $E = 200$ GPa and $I = 100{,}000{,}000$ mm^4, find the slope of the beam at the point of application of the couple.
31. Show that in the case of a simply supported beam, if a couple M is applied at one end, the slope at the point of application of the couple is $Ml/3EI$, where l is the length of the beam.
32. Using Castigliano's theorem, calculate the vertical deflection at the middle of a simply supported beam carrying a uniformly distributed load of w N/m over full span.

33. A solid circular shaft of steel, of diameter 150 mm, is 4 m long, and is subjected to a torque of 10 kN m. Find the strain energy stored in the shaft and the relative rotation between the ends. Take G = 80 GPa.
34. A hollow shaft of external diameter 200 mm and internal diameter 150 mm transmits 1500 kW @ 150 rev/minute. Calculate the strain energy per unit length stored in the shaft and the maximum shear stress. Take G = 85 GPa.
35. A shaft is in two parts, as shown in Fig. 10.67. Find the total strain energy stored in it. Take G = 85 GPa.

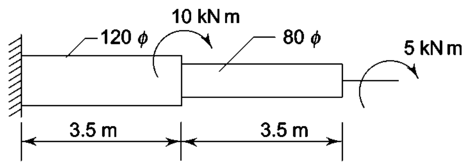


Fig. 10.67

36. The diameter of a shaft, which is 100 mm at one end, reduces to 80 mm at the other end over a length of 500 mm. If it is subjected to a torque of 20 kN m, constant over its length, find the strain energy stored in the shaft. Take G = 80 GPa.
37. Prove that the strain energy of a curved beam with a small initial curvature can be given by $\delta M^2 ds/2EI$.

Solve the following problems using Castigliano's method.

38. Find the deflection and slope at the free end of a cantilever of span 3 m, carrying a uniformly varying from zero at the free end to 25 kN/m at the fixed end. EI is constant.
39. Find the deflection under the loads and slopes at the supports of simply supported beam carrying loads as shown in Fig. 10.68. $EI = 0.8 \times 10^6$ Nm^2.

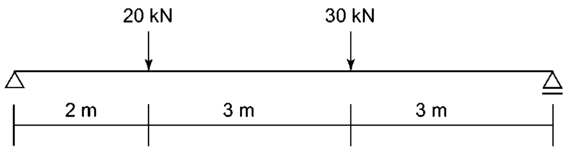


Fig. 10.68

40. Find the deflection under the loads and slopes at the supports of an overhanging beam loaded as shown in Fig. 10.69. EI is constant.

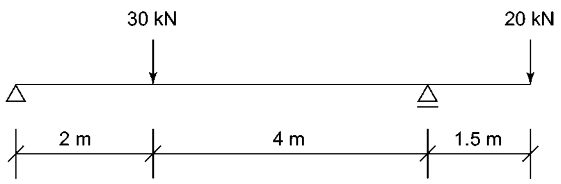


Fig. 10.69

41. Find the horizontal movement of the roller support of the frame loaded as shown in Fig. 10.70.

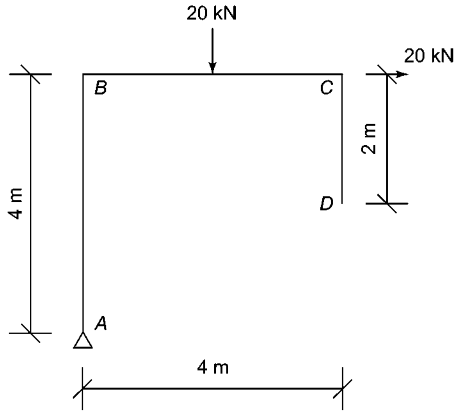


Fig. 10.70

42. Find the horizontal and vertical deflection of the free end of the frame loaded as shown in Fig. 10.71. *EI* is constant.

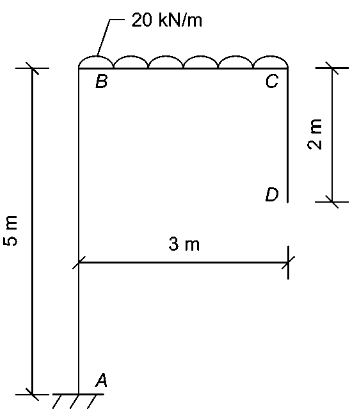


Fig. 10.71

43. The frame shown in Fig. 10.72 carries a single point load of 30 kN at the free end. Find the horizontal and vertical deflection at the free end and the slope at *B*. *EI* is constant.

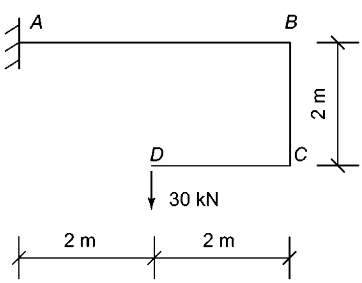


Fig. 10.72

Solve the following problems using the unit load method.

44. A simply supported beam of 9 m span carries a uniformly varying load varying from zero at the left end to 30 kN/m at the right support. Find the slopes at the ends and the deflection at the mid-span.
45. In the overhanging beam loaded as shown in Fig. 10.73, find the slope at support *A* and the deflection at the free end. *EI* is constant.

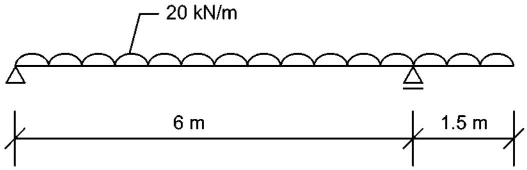


Fig. 10.73

46. Find the horizontal and vertical deflections at the free end *D* of the frame loaded as shown in Fig. 10.74. *EI* is constant.

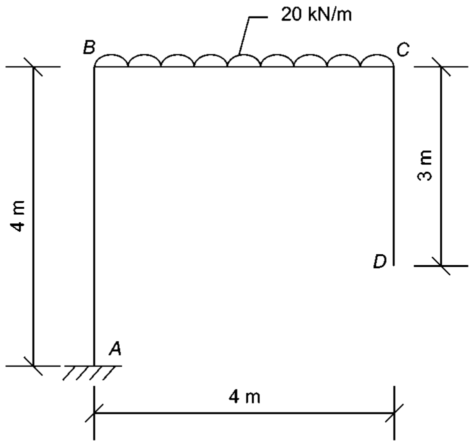


Fig. 10.74

47. Find the horizontal and vertical deflections at the free end *E* of the frame loaded as shown in Fig. 10.75. Take *EI* constant.

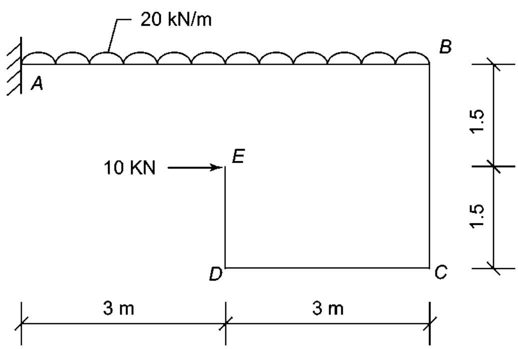


Fig. 10.75

48. In the frame loaded as shown in Fig. 10.76, find the horizontal and vertical deflections at the free end. *EI* is constant.

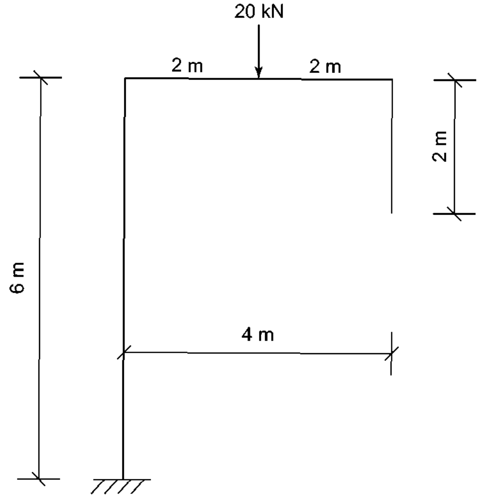


Fig. 10.76

49. In the case of the simply supported beam loaded as shown in Fig. 10.77, find the deflection at the mid-span.

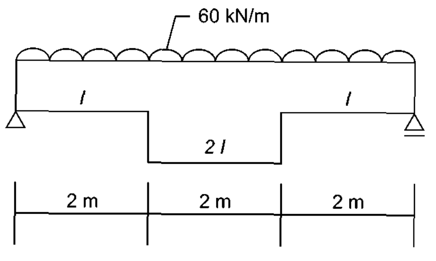


Fig. 10.77