3.2 Castigliano theorem to calculate displacement

The Castigliano theorem allows calculating the displacements u_{xi}, u_{yi} or u_{zi} in a point i respectively along the three axes X, Y or Z of the structure. It is defined as the derivative of the total elastic strain energy with respect to the force F_i applied in this point i:

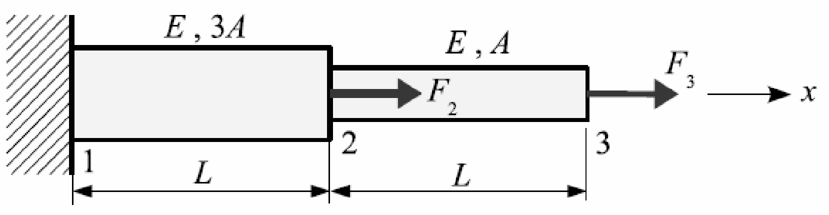
$$u_{xi} = \frac{\partial U^{Total}}{\partial F_i} \tag{III-43}$$

Referring to the figure (Figure III- 4) and knowing that $\sigma = E\varepsilon$, the elastic strain energy in the case of traction or compression loading is equal after indices simplification to:

$$U^{Tensile} = \frac{1}{2}\int_V \sigma\varepsilon \, dV = \frac{1}{2}\int_V \frac{\sigma^2}{E} dV = \frac{1}{2}\int \frac{F^2}{EAA} A dx = \int \frac{F^2}{2EA} dx \tag{III-44}$$

Example 3:

The bar shown in the figure is embedded at 1. Let E be the Young's modulus of the material. The area of the cross section is $3A$ between the points 1 and 2 and A between the points 2 and 3. This bar carries in 2 a force with components (F_2, 0, 0) and in 3 a force with components (F_3, 0, 0) [15].



Let calculate:

1. The expression of the normal force $N(x)$:

$\sum F_{/x} = 0 \Rightarrow Rx - F_2 - F_3 = 0 \Rightarrow Rx = F_2 + F_3$

$0 < x < L \Rightarrow N(x) = N_{12} = Rx = F_2 + F_3$

$L < x < 2L \Rightarrow N(x) = N_{23} = Rx - F_2 = F_2 + F_3 - F_2 = F_3$

Then:

$N_{12} = F_2 + F_3$ and $N_{23} = F_3$

2. The total elastic strain energy U^{Total}:

$$U^{Total} = \int_0^L \frac{N_{12}^2}{2E3A} dx + \int_L^{2L} \frac{N_{23}^2}{2EA} dx = \frac{L}{2EA}\left[\frac{1}{3}(F_2 + F_3)^2 + F_3^2\right]$$

3. The displacements u_{x2} and u_{x3} respectively in the points 2 and 3:

$$u_{x2} = \frac{\partial U^{total}}{\partial F_2} = \frac{L}{3EA}(F_2 + F_3) \; ; \; u_{x3} = \frac{\partial U^{total}}{\partial F_3} = \frac{L}{3EA}(F_2 + 4F_3)$$

4. The flexibility and the stiffness matrices of this bar:

Flexibility matrix $[C]$:

$$\begin{pmatrix} u_{x2} \\ u_{x3} \end{pmatrix} = \frac{L}{3EA}\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}\begin{pmatrix} F_2 \\ F_3 \end{pmatrix} \Rightarrow [C] = \frac{L}{3EA}\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

Rigidity matrix $[K]$:

$$\begin{pmatrix} F_2 \\ F_3 \end{pmatrix} = \frac{EA}{L}\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} u_{x2} \\ u_{x3} \end{pmatrix} \Rightarrow [K] = [C]^{-1} = \frac{EA}{L}\begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

4. Elastic strain energy in bending

In the case of a bending loading that will held in the *XY* plane and around the *Z* axis, we have the bending strain energy due to the normal tensile and compressive stresses generated by the bending moments Mf_z and the shear strain energy induced by the shear forces T_y. The shear strain energy is neglected compared to the bending strain energy. The bending and shear elastic strain energy is equal to:

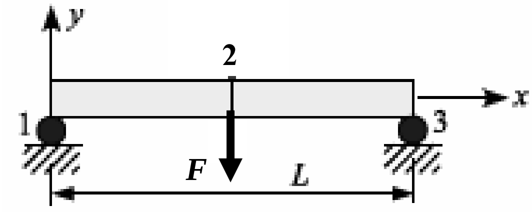
$$U^{Bending} = \frac{1}{2}\int_V \sigma\varepsilon dV + \frac{1}{2}\int_V \tau\gamma dV = \frac{1}{2}\iiint \frac{\sigma^2}{E}dV + \frac{1}{2}\iiint \frac{\tau^2}{\mu}dV \qquad \text{(III-45)}$$

$\sigma = \frac{Mf_z.y}{I_z}$; $\tau = \frac{T_y}{A}$; I_z and A are the inertia moment and the area of the beam cross section.

$$\begin{aligned} U^{Bending} &= \frac{1}{2}\iiint \frac{Mf_z^2 y^2}{EI_z^2}dsdx + \frac{1}{2}\int \frac{T_y^2}{\mu A^2}Adx \\ &= \frac{1}{2}\int \frac{Mf_z^2}{EI_z^2}dx\iint y^2 ds + \frac{1}{2}\int \frac{T_y^2}{\mu A}dx \\ &= \frac{1}{2}\int \frac{Mf_z^2 I_z}{EI_z^2}dx + \frac{1}{2}\int \frac{T_y^2}{\mu A}dx \qquad \text{(III-46)} \\ &= \frac{1}{2}\int \frac{Mf_z^2}{EI_z}dx + \frac{1}{2}\int \frac{T_y^2}{\mu A}dx \end{aligned}$$

Example 1:

The following beam of length L and constant section (quadratic moment: I_z) is supported at 1 and 3 on a simple support. The beam is made with steel having a **Young's** modulus E. It carries in its center (middle point 2) a force with components (0, F, 0).



We neglect the influence of the shear force: Bernoulli model. Let calculate:

1. The expression of the bending moment $Mf_z(x)$:

$$0 < x < \frac{L}{2} \Rightarrow Mf_1(x) = Mf_{12} = \frac{F}{2}x$$

$$\frac{L}{2} < x < L \Rightarrow Mf_2(x) = Mf_{23} = \frac{F}{2}(L-x)$$

2. The total elastic strain energy U^{Total} :

$$U^{Total} = \frac{1}{2EI_z}\left(\int_0^{L/2} Mf_{12}^2 dx + \int_{L/2}^{L} Mf_{23}^2 dx\right) = \frac{F^2}{8EI_z}\left(\int_0^{L/2} x^2 dx + \int_{L/2}^{L} (L-x)^2 dx\right)$$

$$= \frac{F^2}{8EI_z}\left(\frac{1}{3}x^3\Big|_0^{L/2} - \frac{1}{3}(L-x)^3\Big|_{L/2}^{L}\right)$$

$$U^{Total} = \frac{F^2L^3}{96EI_z}$$

3. The deflection in point 2:

$$u_{y2} = \frac{\partial U^{total}}{\partial F} = \frac{FL^3}{48EI_z}$$

$\Rightarrow F = \frac{48EI_z}{L^3} \times u_{y2} \Rightarrow$ The bending rigidity is equal to $\frac{48EI_z}{L^3}$; it depends proportionally to E and I_z and inversely to L.

4. Comparison of the deflection of the point 2 calculated with Castigliano theorem with the deflection calculated by the double-integration method:

Using the double-integration method seen in the Chapter II, the deflection *y(x)* for this example is equal to:

$$y_1(x) = \frac{1}{4EI_z}\left(\frac{F}{3}x^3 - \frac{FL^2}{4}x\right)$$

Now, the maximum deflection f_{max} is located in the center of the beam, and then f_{max} is equal to:

$$f_{\max} = y_1(L/2) = -\frac{FL^3}{48EI_z} \Rightarrow u_{y2} = |f_{\max}|$$

We find the same expression obtained by the Castigliano theorem

5. Elastic strain energy in torsion

5.1 Torsion rigidity and shearing stress induced by torsion

The twist angle θ must be less than 1° over a shaft length equal to 20 times the shaft diameter. The example presented in the below figure shows a shaft of diameter *d*, radius *R* and length *L* subjected to a torque M_t.

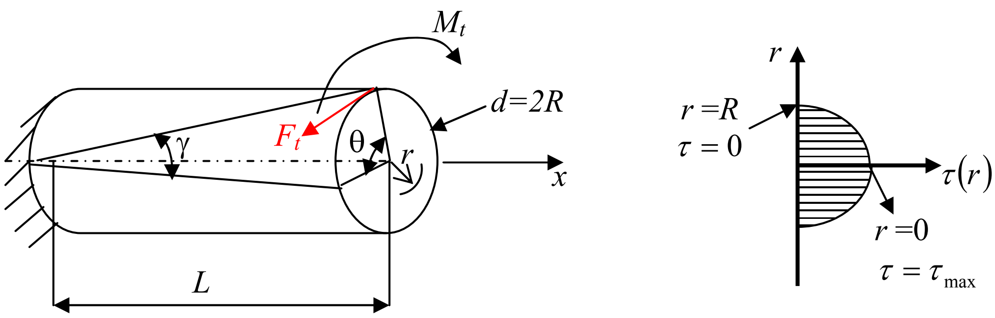


Figure III- 7: Torsion loading and its generated shearing stress

We know that:

$$dM_t = \tau(r) \times r \times ds \tag{III-47}$$

$\tau(r)$ is the shear stress at a point on the cross-section area A which is located at a radius r from the center of this area A. ds is a small area of the total area A. Thus, $\tau(r)$ is equal to:

$$\tau(r) = \mu \times \gamma(r) = \mu \times r \times \frac{d\theta}{dx} \quad \text{with} \quad \gamma(r) = \frac{rd\theta}{dx} = \frac{R\theta}{L} \tag{III-48}$$

$$\text{(III-47)} \Rightarrow \int dM_t = \int \mu \times r \times \frac{d\theta}{dx} \times r \times ds$$

$$\Rightarrow M_t = \mu \times \frac{d\theta}{dx} \times \int r^2 ds = \mu \times \frac{d\theta}{dx} \times I_p \tag{III-49}$$

$$\Rightarrow \int_0^L M_t dx = \int_0^\theta \mu \times I_p \times d\theta$$

After integration we can write:

$$M_t = \frac{\mu \times I_p}{L}\theta = k_t.\theta = F_t \times R \tag{III-50}$$

k_t is the torsional rigidity in N.m, it is equal to:

$$K_{tors} = \frac{\mu \times I_P}{L} \tag{III-51}$$

The elastic shear modulus μ is equal to:

$$\mu = \frac{E}{2(\nu + 1)} \tag{III-52}$$

The polar inertia moment I_p is calculated by the following formula:

$$I_p = \int r^2 ds = \frac{\pi d^4}{32} = \frac{\pi R^4}{2} \tag{III-53}$$

The twist angle θ in radians is calculated like this:

$$\theta = \frac{M_t.L}{\mu \times I_p} \tag{III-54}$$

5.2 Torsion strain energy

We know that from equation (III-48):

$$\tau(r) = \mu \times r \times \frac{d\theta}{dx} \tag{III-55}$$

And from equation (III-49), we have:

$$\frac{d\theta}{dx} = \frac{M_t}{\mu \times I_p} \tag{III-56}$$

Substituting (III-56) into (III-55), we obtain:

$$\tau(r) = \mu \times r \times \frac{M_t}{\mu \times I_p}$$
$$\Rightarrow \tau(r) = \frac{M_t \times r}{I_p} \tag{III-57}$$

$$\Rightarrow \gamma(r) = \frac{\tau(r)}{\mu} = \frac{M_t \times r}{\mu \times I_p} \tag{III-58}$$

The torsional strain energy is given by:

$$U^{Torsion} = \frac{1}{2}\int_V \tau\gamma dV = \frac{1}{2}\iiint \frac{\tau^2}{\mu} dV$$
$$U^{Torsion} = \frac{1}{2}\iiint \frac{M_t^2 r^2}{\mu \times I_p^{\,2}} ds dx = \frac{1}{2}\int \frac{M_t^2}{\mu \times I_p^{\,2}} dx \iint r^2 ds = \frac{1}{2}\int \frac{M_t^2 I_p}{\mu \times I_p^{\,2}} dx \tag{III-59}$$
$$U^{Torsion} = \frac{1}{2}\int \frac{M_t^2}{\mu \times I_p} dx$$

The deformation energy stored in a shaft with a length *L* can also be equal to:

$$U^{Torsion} = \frac{1}{2} k_t . \theta^2 \tag{III-60}$$

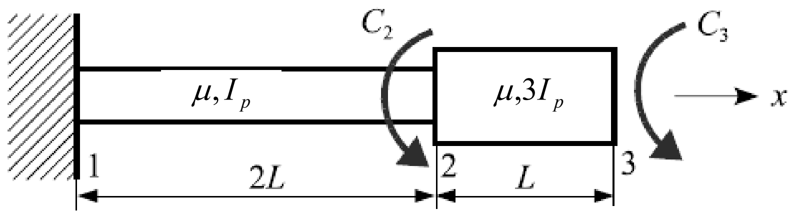
5.3 Castigliano theorem to calculate rotation

The Castigliano theorem allows calculating the rotations θ_{xi}, θ_{yi} or θ_{zi} in a point *i* respectively around the three axes *X*, *Y* or *Z* of the structure. It is defined as the derivative of the total elastic strain energy with respect to the moment M_i applied in this point *i*:

$$\theta_{xi} = \frac{\partial U^{Total}}{\partial M_i} \tag{III-61}$$

<u>Example 1:</u>

We consider the x-axis shaft shown in the below figure. Let μ be the transverse modulus of elasticity of the shaft material. The torsion constant is I_p between the points 1 and 2 and $3I_p$ between the points 2 and 3. The point 1 is embedded; the points 2 and 3 carry a respective torque intensities C_2 and C_3 (see the below figure). Let calculate:



1. The expression of the torsion torque $M_t(x)$:

$$\sum M_{/x} = 0 \Rightarrow M_{tx} - C_2 - C_3 = 0 \Rightarrow M_{tx} = C_2 + C_3$$

$$0 < x < 2L \Rightarrow M_t^{12} = M_{tx} = C_2 + C_3$$

$$2L < x < 3L \Rightarrow M_t^{23} = M_{tx} - C_2 = C_2 + C_3 - C_2 = C_3$$

Then:

$M_t^{12} = C_2 + C_3$ and $M_t^{23} = C_3$

2. The total elastic strain energy U^{Total}:

$$U^{Total} = \int_0^{2L} \frac{\left(M_t^{12}\right)^2}{2\mu I_p} dx + \int_{2L}^{3L} \frac{\left(M_t^{23}\right)^2}{2\mu 3 I_p} dx = \frac{L}{\mu I_p}\left[\left(C_2 + C_3\right)^2 + \frac{1}{6}C_3^2\right]$$

3. The rotations θ_{x2} and θ_{x3} respectively in the points 2 and 3:

$$\theta_{x2} = \frac{\partial U^{total}}{\partial C_2} = \frac{L}{\mu I_p}\left(2C_2 + 2C_3\right) \; ; \; \theta_{x3} = \frac{\partial U^{total}}{\partial C_3} = \frac{L}{\mu I_p}\left(2C_2 + \frac{7}{3}C_3\right)$$

4. The flexibility and the stiffness matrices of this shaft:

Flexibility matrix $[C]$:

$$\begin{pmatrix} \theta_{x2} \\ \theta_{x3} \end{pmatrix} = \frac{L}{\mu I_p}\begin{bmatrix} 2 & 2 \\ 2 & 7/3 \end{bmatrix}\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} \Rightarrow [C] = \frac{L}{\mu I_p}\begin{bmatrix} 2 & 2 \\ 2 & 7/3 \end{bmatrix} = \frac{L}{3\mu I_p}\begin{bmatrix} 6 & 6 \\ 6 & 7 \end{bmatrix}$$

Rigidity matrix $[K]$:

$$\begin{pmatrix} C_2 \\ C_3 \end{pmatrix} = \frac{\mu I_p}{2L}\begin{bmatrix} 7 & -6 \\ -6 & 6 \end{bmatrix}\begin{pmatrix} \theta_{x2} \\ \theta_{x3} \end{pmatrix} \Rightarrow [K] = [C]^{-1} = \frac{\mu I_p}{2L}\begin{bmatrix} 7 & -6 \\ -6 & 6 \end{bmatrix}$$

6. General expression of elastic strain energy

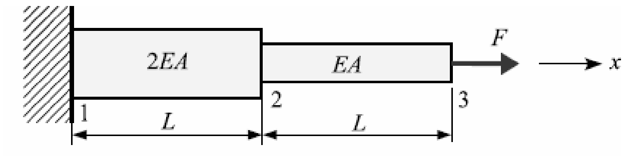
In the case of a combined loadings, the global elastic strain energy for all the loadings (traction, compression, shearing, bending and torsion) is equal to:

$$U^{Global} = \frac{1}{2}\left(\int \frac{F^2}{EA}dx + \int \frac{Mf_z^{\,2}}{EI_z}dx + \int \frac{T_y^{\,2}}{\mu A}dx + \int \frac{Mf_y^{\,2}}{EI_y}dx + \int \frac{T_z^{\,2}}{\mu A}dx + \int \frac{Mt^2}{\mu I_p}dx\right) \qquad \text{(III-62)}$$

Directed works No. 3 "Energetic methods for elastic systems"

Exercise N°1

The beam shown in the figure below is embedded at 1. Let E be the Young's modulus of the material. The area of the cross section is 2A between points 1 and 2 and A between points 2 and 3. The beam carries in 3 a force with components (F, 0, 0).



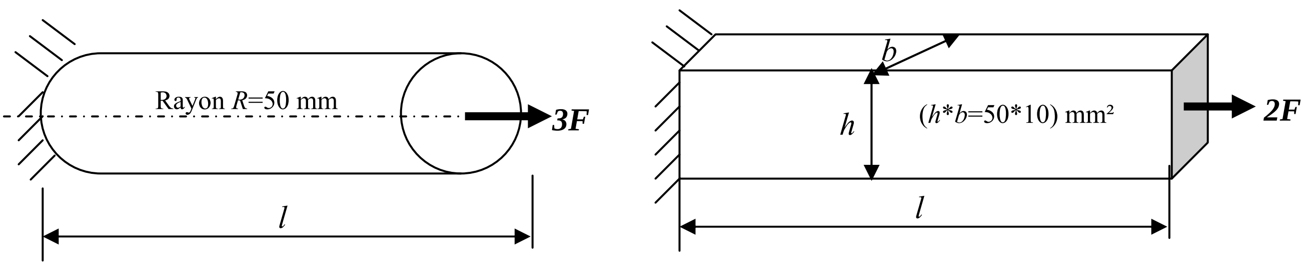
1) Determine the expression for the normal force N(x).

2) If E=200 GPa, A=40mm², L=200 mm and the force F=100 N, then calculate the elastic strain energy and the displacement u_3.

Exercise N°2

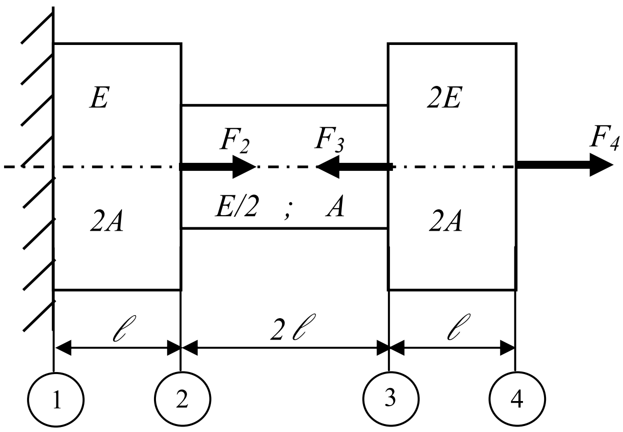
Two beams with circular and rectangular cross-sections embedded on their left ends respectively undergo an extension on their right ends by forces with a magnitude of 3F and 2F.

1) If F= 70daN, the length l=2m and the Young's modulus of the material of the two beams E= 210 GPa, calculate the strain energy that the two beams will undergo as well as the displacement of their right ends.



Exercise N°3

For the example of a bar shown in the below figure, establish the expression of the total strain energy $E_{Déf}^{Total}$ and calculate the displacement u_4 if F_2 = 20kN ; F_3 = 20daN ; F_4 = 20N ; ℓ = 2m ; E = 300 GPa ; A = 10 cm².

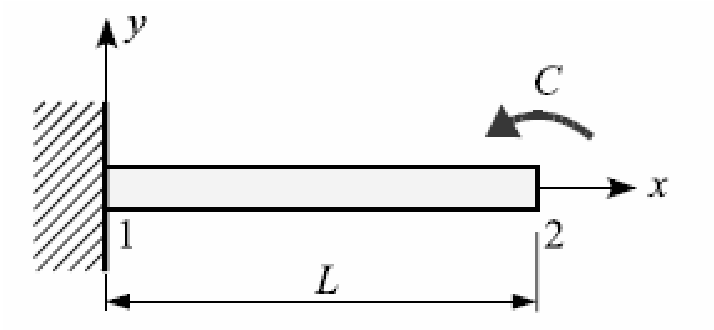


Exercise N°4

The beam below of length L and of constant circular cross section with a diameter d, the beam is embedded at the left point 1. It is made of steel with Young's modulus E. It carries at 2 a torque around the z axis with components (0, 0, C). By neglecting the influence of the shear force (according to the Bernoulli model):

1) Determine the expression for the bending moment *Mf(x)*.

2) Calculate the elastic strain energy knowing that E=200000 MPa, d=30mm, L=0.2 m and the torque C=100 N.m and thus calculate the rotation θ_2.



Exercise N°5

The beam presented in the below figure has a length L and a constant circular cross section with a diameter d, the beam is embedded at the left point 1. It is made of steel with Young's modulus E. It carries at 2 a torque with components (0, 0, C) and a force of components (0, F, 0). Neglecting the influence of the shear force, determine:

1) The expression for the bending moment *Mf(x)*.

2) The elastic strain energy knowing that E=200000 MPa, d=30mm, L=0.2 m, F=50 N and the torque C=100 N.m.

3) Using Castigliano's theorem, determine the deflection of the beam at the point 2.