This expression can also be used when some of the *normal* stresses are compressive, in which case they must be given a negative sign.

If in addition to normal stresses there are shearing stresses acting on the faces of the element, the energy of shear can be added to the energy of tension or compression (see p. 299), and using eq. (179) the total energy stored in one cubic inch is

$$w = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\mu}{E}(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2). \quad (193)$$

As a second example let us consider a beam supported at the ends, loaded at the middle by a force P and bent by a couple M applied at the end A. The deflection at the middle is, from eqs. (90) and (105),

$$\delta = \frac{Pl^3}{48EI} + \frac{Ml^2}{16EI}. \quad (a)$$

The slope at the end A is, from eqs. (88) and (104),

$$\theta = \frac{Pl^2}{16EI} + \frac{Ml}{3EI}. \quad (b)$$

Then the strain energy of the beam, equal to the work done by the force P and by the couple M, is

$$U = \frac{P\delta}{2} + \frac{M\theta}{2} = \frac{1}{EI}\left(\frac{P^2l^3}{96} + \frac{M^2l}{6} + \frac{MPl^2}{16}\right). \quad (c)$$

This expression is a homogeneous function of the second degree in the external force and the external couple. Solving eqs. (*a*) and (*b*) for M and P and substituting in eq. (*c*), an expression for the strain energy in the form of a homogeneous function of the second degree in displacements may be obtained. It must be noted that when external couples are acting on the body the corresponding displacements are the angular displacements of surface elements on which these couples are acting.

69. The Theorem of Castigliano.—From the expressions for the energy of strain in various cases a very simple method for calculating the displacements of points of an elastic body during deformation may be established. For example, in the case of simple tension (Fig. 1), the strain energy as given by eq. (168) is

$$U = \frac{P^2l}{2AE}.$$

By taking the derivative of this expression with respect to P we obtain

$$\frac{dU}{dP} = \frac{Pl}{AE} = \delta,$$

i.e., the derivative of the strain energy with respect to the load gives the displacement *corresponding* to the load, i.e., at the point of application of the load in the direction of the load. In the case of a cantilever loaded at the end, the strain energy is (eq. *c*, p. 297)

$$U = \frac{P^2l^3}{6EI}.$$

The derivative of this expression with respect to the load P gives the known deflection at the free end $Pl^3/3EI$.

In the twist of a circular shaft the strain energy is (eq. 182)

$$\frac{M_t^2l}{2GI_p}.$$

The derivative of this expression with respect to the torque gives

$$\frac{dU}{dM_t} = \frac{M_tl}{GI_p} = \varphi,$$

which is the angle of twist of the shaft, and represents the displacement *corresponding* to the torque.

When several loads act on an elastic body, the same method of calculation of displacements may be used. For example, expression (*c*) of the previous article gives the strain energy of a beam bent by a load P at the middle and by a couple M at the end. The partial derivative of this expression with respect to P gives the deflection under the load and the partial derivative with respect to M gives the angle of rotation of the end of the beam on which the couple M acts.

The theorem of Castigliano is a general statement of these results.[20] If the material of the system follows Hooke's law and the conditions are such that the small displacements due to deformation can be neglected in discussing the action of forces, the strain energy of such a system may be given by a homogeneous function of the second degree in the acting forces (see art. 68). Then the partial derivative of strain energy with respect to any such force gives the displacement corresponding to this force (exceptional cases see art. 72). The terms "force" and "displacement" here may have their generalized meanings, that is, they include "couple" and "angular displacement" respectively.

Let us consider a general case such as shown in Fig. 252. Assume that the strain energy is represented as a function of the forces $P_1, P_2, P_3, \cdots$, so that

$$U = f(P_1, P_2, P_3, \cdots). \tag{a}$$

If a small increase dP_n is given to any external load P_n, the strain energy will increase also and its new amount will be

$$U + \frac{\partial U}{\partial P_n} dP_n. \tag{b}$$

But the magnitude of the strain energy does not depend upon the order in which the loads are applied to the body—it depends only upon their final values. It can be assumed, for instance, that the infinitesimal load dP_n was applied first, and afterwards the loads $P_1, P_2, P_3, \cdots$. The final amount of strain energy remains the same, as given by eq. (*b*). The load dP_n, applied first, produces only an infinitesimal displacement, so that the corresponding work done is a small quantity of the second order and can be neglected. Applying now the loads $P_1, P_2, P_3, \cdots$, it must be noticed that their effect will not be

[20] See the paper by Castigliano, "Nuova Teoria Intorno dell' Equilibrio dei Sistemi Elastici," Atti della Academia delle scienze, Torino, 1875. See also his "Théorie de l'équilibre des systèmes élastiques," Turin, 1879. For an English translation of Castigliano's work see E. S. Andrews, London, 1919.

modified by the load dP_n previously applied[21] and the work done by these loads will be equal to U (eq. *a*), as before. But during the application of these forces, however, dP_n is given some displacement δ_n in the direction of P_n, and does the work $(dP_n)\delta_n$. The two expressions for the work must be equal; therefore

$$U + \frac{\partial U}{\partial P_n}(dP_n) = U + (dP_n)\delta_n,$$

$$\delta_n = \frac{\partial U}{\partial P_n}. \tag{194}$$

As an application of the theorem let us consider a cantilever beam carrying a load P and a couple M_a at the end, Fig. 253. The bending moment at a cross section mn is $M = -Px - M_a$ and the strain energy, from equation (184), is

$$U = \int_0^l \frac{M^2dx}{2EI}.$$

To obtain the deflection δ at the end of the cantilever we have only to take the partial derivative of U with respect to P, which gives

$$\delta = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial P}dx.$$

Substituting for M its equivalent expression, in terms of P and M_a, we obtain

$$\delta = \frac{1}{EI}\int_0^l (Px + M_a)xdx = \frac{Pl^3}{3EI} + \frac{M_a l^2}{2EI}.$$

The same expression would have been obtained by applying one of the previously described methods, such as the area moment method.

To obtain the slope at the end we calculate the partial

[21] This follows from the provisions made on page 306 on the basis of which the strain energy was obtained as a homogeneous function of the second degree.

derivative of the strain energy with respect to the couple M_a. Then

$$\theta = \frac{\partial U}{\partial M_a} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial M_a}dx$$

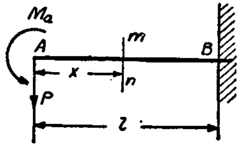

FIG. 253.

$$= \frac{1}{EI}\int_0^l (Px + M_a)dx = \frac{Pl^2}{2EI} + \frac{M_a l}{EI}.$$

The positive signs obtained for δ and θ indicate that the deflection and rotation of the end have the same directions respectively as the force and the couple in Fig. 253.

It should be noted that the partial derivative $\partial M/\partial P$ is the rate of increase of the moment M with respect to the increase of the load P and can be visualized by the bending moment diagram for a load equal to unity, as shown in Fig. 254 (*a*). The partial derivative $\partial M/\partial M_a$ can be visualized

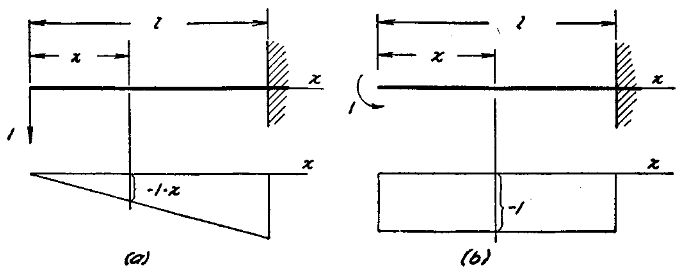

FIG. 254.

in the same manner by the bending moment diagram in Fig. 254 (*b*). Using the notations

$$\frac{\partial M}{\partial P} = M_p' \quad \text{and} \quad \frac{\partial M}{\partial M_a} = M_m'$$

we can represent our previous results in the following form:

$$\delta = \frac{1}{EI}\int_0^l MM_p'dx; \qquad \theta = \frac{1}{EI}\int_0^l MM_m'dx. \quad (195)$$

These equations, derived for the particular case shown in Fig. 253, also hold for the general case of a beam with any kind of loading and any kind of support. They can also be used in the case of distributed loads.

Let us consider, for example, the case of a uniformly loaded and simply supported beam, Fig. 255, and calculate the deflection at the middle of this beam by using the Castigliano theorem. In the preceding cases concentrated forces and couples acted, and partial derivatives with respect to these forces and couples gave the corresponding displacements and

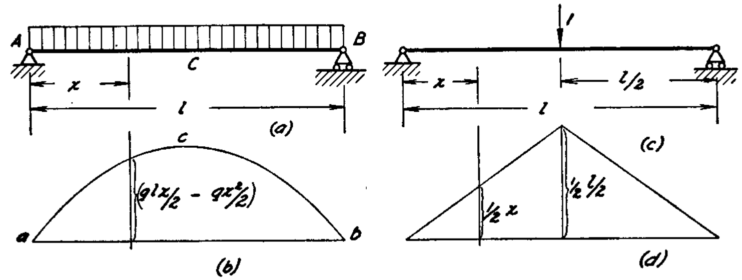

FIG. 255.

rotations. In the case of a uniform load there is no vertical force acting at the middle of the beam which would *correspond* to the deflection at the middle. Thus we cannot proceed as in the previous problem. This difficulty can, however, be readily removed by assuming that there is a fictitious load P of infinitely small magnitude at the middle. Such a force evidently will not affect the deflection and the bending moment diagram shown in Fig. 255 (*b*). At the same time, the rate of increase of the bending moment due to the increase of P, represented by the partial derivative $\partial M/\partial P$, is as represented by Fig. 255 (*c*) and 255 (*d*). With these values of M and $\partial M/\partial P$ the value of the deflection is

$$\delta = \frac{\partial U}{\partial P} = \frac{1}{EI_z}\int_0^l M\frac{\partial M}{\partial P}dx.$$

Observing that M and $\partial M/\partial P$ are both symmetrical with respect to the middle of the span, we obtain

$$\delta = \frac{2}{EI_z}\int_0^{l/2} M \frac{\partial M}{\partial P}\,dx$$

$$= \frac{2}{EI_z}\int_0^{l/2}\left(\frac{qlx}{2} - \frac{qx^2}{2}\right)\frac{x}{2}\,dx = \frac{5}{384}\frac{ql^4}{EI_z}.$$

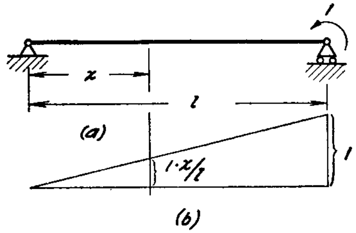

FIG. 256.

If it is required to calculate the slope at the end B of the beam in Fig. 255 (a) by using the Castigliano theorem, we have only to assume an infinitely small couple M_b applied at B. Such a couple does not change the bending moment diagram in Fig. 255 (b). The partial derivative $\partial M/\partial M_b$ is then as represented in Fig. 256 (a) and 256 (b). The required rotation of the end B of the beam is then

$$\theta = \frac{\partial U}{\partial M_b} = \frac{1}{EI_z}\int_0^{l} M \frac{\partial M}{\partial M_b}\,dx$$

$$= \frac{1}{EI_z}\int_0^{l}\left(\frac{qlx}{2} - \frac{qx^2}{2}\right)\frac{x}{l}\,dx$$

$$= \frac{ql^3}{24EI_z}.$$

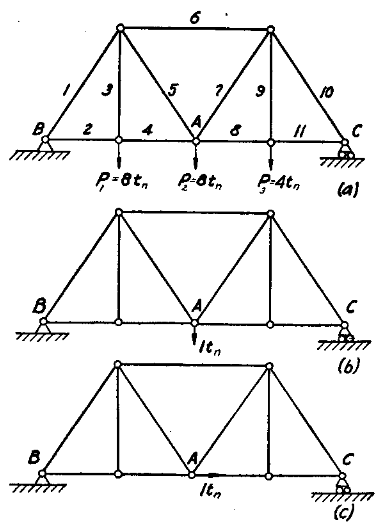

FIG. 257.

We see that the results obtained by the use of Castigliano's theorem coincide with those previously obtained (p. 138).

The Castigliano theorem is especially useful in the calculation of deflections in trusses. As an example let us consider the case shown in Fig. 257. All members of the system are numbered and their lengths and cross-sectional areas given in the table below. The force S_i produced in any bar i of the system by the loads P_1, P_2, P_3

may be calculated from simple equations of statics. These forces are given in column 4 of the table. The strain energy

TABLE OF DATA FOR THE TRUSS IN FIGURE 257

1	2	3	4	5	6	7
i	l_i in.	A_i in.^{2}	S_i tn.	S_i'	$\frac{S_iS_i'l_i}{A_i}$	S_i''
1	250	6	−13.75	−0.625	358	0
2	150	3	8.25	0.375	155	1
3	200	2	8.00	0	0	0
4	150	3	8.25	0.375	155	1
5	250	2	3.75	0.625	293	0
6	300	4	−10.50	−0.750	59	0
7	250	2	6.25	0.625	488	0
8	150	3	6.75	0.375	127	0
9	200	2	4.00	0	0	0
10	250	6	−11.25	−0.625	293	0
11	150	3	6.75	0.375	127	0

$$\sum_{i=1}^{i=m}\frac{S_iS_i'l_i}{A_i} = 2{,}055 \text{ tns. per inch.}$$

of any bar i, from eq. (168), is $S_i^2l_i/2A_iE$. The amount of strain energy in the whole system is

$$U = \sum_{i=1}^{i=m}\frac{S_i^2l_i}{2A_iE}, \qquad (196)$$

in which the summation is extended over all the members of the system, which in our case is $m = 11$. The forces S_i are functions of the loads P, and the deflection δ_n under any load P_n is, therefore, from eq. (194),

$$\delta_n = \frac{\partial U}{\partial P_n} = \sum_{i=1}^{i=m}\frac{S_il_i}{A_iE}\cdot\frac{\partial S_i}{\partial P_n}. \qquad (197)$$

The derivative $\partial S_i/\partial P_n$ is the rate of increase of the force S_i with increase of the load P_n. Numerically it is equal to the force produced in the bar i by a unit load applied in the position of P_n, and we will use this fact in finding the above derivative. These derivatives will hereafter be denoted by

S_i'. The equation for calculating the deflections then becomes

$$\delta_n = \sum_{i=1}^{i=m} \frac{S_i S_i' l_i}{A_i E}. \tag{198}$$

Consider for instance the deflection δ_2 corresponding to P_2 at A in Fig. 257 (*a*). The magnitudes S_i' tabulated in column 5 above are obtained by the simple principles of statics from the loading conditions shown in Fig. 257 (*b*), in which all actual loads are removed and a vertical load of one ton is applied at the hinge A. The values tabulated in column 6 are calculated from those entered in columns 2 through 5. Summation and division by the modulus $E = (30/2{,}000) \times 10^6$ tns. per sq. in. gives the deflection at A, eq. (198),

$$\delta_2 = \frac{2{,}055 \times 2{,}000}{30 \times 10^6} = 0.137 \text{ in.}$$

The above discussion was concerned with the computation of displacements $\delta_1, \delta_2, \cdots$ corresponding to the given external forces $P_1, P_2, \cdots$. In investigating the deformation of an elastic system, it may be necessary to calculate the displacement of a point at which there is no load at all, or the displacement of a loaded point in a direction different from that of the load. The method of Castigliano may also be used here. We merely apply at that point an additional infinitely small *imaginary load* Q in the direction in which the displacement is wanted, and calculate the derivative $\partial U/\partial Q$. In this derivative the added load Q is put equal to zero, and the desired displacement obtained. For example, in the truss shown in Fig. 257 (*a*), let us calculate the horizontal displacement of the point A. A horizontal force Q is applied at this point, and the corresponding horizontal displacement is

$$\delta_h = \left(\frac{\partial U}{\partial Q}\right)_{Q=0} = \sum_{i=1}^{i=m} \frac{S_i l_i}{A_i E} \cdot \frac{\partial S_i}{\partial Q}, \tag{d}$$

in which the summation is extended over all the members of the system. The forces S_i in eq. (*d*) have the same meaning as before, because the added load Q is zero, and the derivatives $\partial S_i/\partial Q = S_i''$ are obtained as the forces in the bars of the truss produced by the loading shown in Fig. 257 (*c*). These are tabulated in column 7. Substituting these forces into eq. (*d*), we find that the horizontal displacement of A is equal to the sum of the elongations of the bars 2 and 4, namely,

$$\delta_h = \frac{1}{E}\left(\frac{S_2 l_2}{A_2} + \frac{S_4 l_4}{A_4}\right) = \frac{150 \times 2{,}000}{3 \times 30 \times 10^6}(8.25 + 8.25) = 0.055 \text{ in.}$$

In investigating the deformation of trusses it is sometimes necessary to know the change in distance between two points of the system. This can also be done by the Castigliano method. Let us determine, for instance, what decrease δ, in the distance between the joints A and B (Fig. 258, *a*), is produced by the loads P_1, P_2, P_3. At these joints, two equal and opposite imaginary forces Q are applied as indicated in the figure by the dotted lines. It follows from the Castigliano theorem that the partial derivative $(\partial U/\partial Q)_{Q=0}$ gives the sum of the displacements of A and B, in the direction AB, produced by the loads P_1, P_2, P_3. Using eq. (194), this displacement is[22]

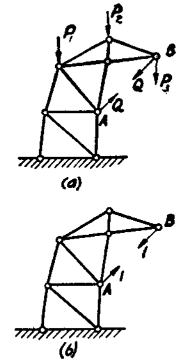

Fig. 258.

$$\delta = \left(\frac{\partial U}{\partial Q}\right)_{Q=0} = \sum_{i=1}^{i=m} \frac{S_i l_i}{A_i E} \frac{\partial S_i}{\partial Q_i} = \sum_{i=1}^{i=m} \frac{S_i l_i}{A_i E} \cdot S_i', \tag{199}$$

in which S_i are the forces produced in the bars of the system by the actual loads P_1, P_2, P_3; S_i' are the quantities to be determined from the loading shown in Fig. 258 (*b*), in which all actual loads are removed and two opposite unit forces are applied at A and B; and m is the number of members.

Problems

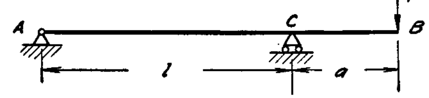

Fig. 259.

1. Determine by the use of Castigliano's theorem the deflection and the slope at the end of a uniformly loaded cantilever beam.

2. Determine the deflection at the end B of the overhang of the beam shown in Fig. 259.

[22] This problem was first solved by J. C. Maxwell, "On the Calculation of the Equilibrium and Stiffness of Frames," Phil. Mag. (4), Vol. 27, 1864, p. 294. Scientific Papers, Vol. 1, Cambridge, 1890, p. 598.

3. A system consisting of two prismatical bars of equal length and equal cross section (Fig. 260) carries a vertical load P. Determine the vertical displacement of the hinge A.

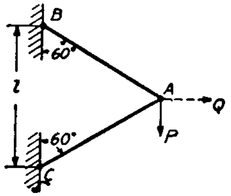

FIG. 260.

Solution. The tensile force in the bar AB and compressive force in the bar AC are equal to P. Hence the strain energy of the system is

$$U = 2\frac{P^2l}{2AE}.$$

The vertical displacement of A is

$$\delta = \frac{dU}{dP} = \frac{2Pl}{AE}.$$

4. Determine the horizontal displacement of the hinge A in the previous problem.

Solution. Apply a horizontal imaginary load Q as shown in Fig. 260 by the dotted line. The potential energy of the system is

$$U = \frac{(P + 1/\sqrt{3}Q)^2l}{2AE} + \frac{(P - 1/\sqrt{3}Q)^2l}{2AE}.$$

The derivative of this expression with respect to Q for $Q = 0$ gives the horizontal displacement

$$\delta_h = \left(\frac{\partial U}{\partial Q}\right)_{Q=0} = \left(\frac{2Ql}{3AE}\right)_{Q=0} = 0.$$

5. Determine the angular displacement of the bar AB produced by the load P in Fig. 261.

Solution. An imaginary couple M is applied to the system as shown in the figure by dotted lines. The displacement corresponding to this couple is the angular displacement φ of the bar AB due to the load P. The forces S_i in this case are: $P + 1/\sqrt{3}(M/l)$ in the bar AB and $-P - 2/\sqrt{3}(M/l)$ in the bar AC. The strain energy is

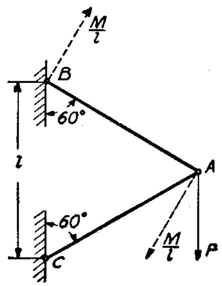

FIG. 261.

$$U = \frac{l}{2AE}\left[\left(P + 1/\sqrt{3}\frac{M}{l}\right)^2 + \left(-P - 2/\sqrt{3}\frac{M}{l}\right)^2\right],$$

from which

$$\varphi = \left(\frac{\partial U}{\partial M}\right)_{M=0} = \left(\frac{P\sqrt{3}}{AE} + \frac{5M}{3lAE}\right)_{M=0} = \frac{P\sqrt{3}}{AE}.$$

6. What horizontal displacement of the support B of the frame shown in Fig. 262 is produced by horizontal force H?

Answer.

$$\delta_h = \frac{2}{3}\frac{Hh^3}{EI_1} + \frac{Hh^2l}{EI}.$$

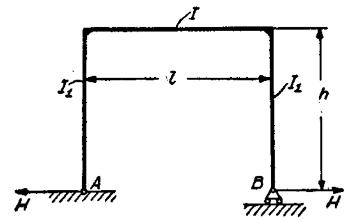

FIG. 262.

7. Determine the vertical displacement of the point A and horizontal displacement of the point C of the steel truss shown in Fig. 263 if $P = 2{,}000$ lbs., the cross-sectional areas of the compressed bars are 5 sq. in., and of the other bars 2 sq. in.

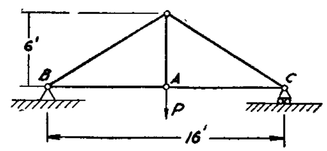

FIG. 263.

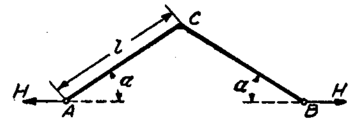

FIG. 264.

8. Determine the increase in the distance AB produced by forces H (Fig. 264) if the bars AC and BC are of the same dimensions and only the bending of the bars need be taken into account. It is assumed that α is not small, so that the effect of deflections on the magnitude of bending moment can be neglected.

Answer.

$$\delta = \frac{2}{3}\frac{H\sin^2\alpha\, l^3}{EI}.$$

9. Determine the deflection at a distance a from the left end of the uniformly loaded beam shown in Fig. 255 (a).

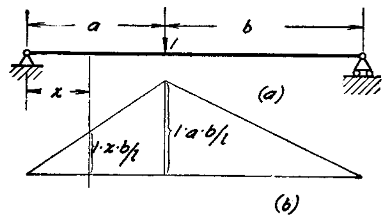

FIG. 265.

Solution. Applying an infinitely small load P at a distance a from the left end, the partial derivative $\partial M/\partial P$ is as visualized in Fig. 265 (a) and 265 (b). Using for M the parabolic diagram in Fig. 255 (b), the desired deflection is

$$\delta = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^l M\frac{\partial M}{\partial P}dx = \frac{1}{EI}\int_0^a\left(\frac{qlx}{2} - \frac{qx^2}{2}\right)\frac{xb}{l}dx$$
$$+ \frac{1}{EI}\int_b^l\left(\frac{qlx}{2} - \frac{qx^2}{2}\right)\frac{a(l-x)}{l}dx = \frac{qab}{24EI}(a^2 + b^2 + 3ab).$$

Substituting x for a and $l - x$ for b this result can be brought into agreement with the equation for the deflection curve previously obtained (p. 138).

70. Application of Castigliano Theorem in Solution of Statically Indeterminate Problems.—The Castigliano theorem is very useful also in the solution of statically indeterminate problems. Let us begin with problems in which the reactions at the supports are considered as the statically indeterminate quantities. Denoting by X, Y, Z, $\cdots$ the statically indeterminate reactive forces, the strain energy of the system is a function of these forces. For the immovable supports and for the supports whose motion is perpendicular to the direction of the reactions the partial derivatives of the strain energy with respect to the unknown reactive forces must be equal to zero by the Castigliano theorem. Hence

$$\frac{\partial U}{\partial X} = 0; \quad \frac{\partial U}{\partial Y} = 0; \quad \frac{\partial U}{\partial Z} = 0; \quad \cdots. \tag{200}$$

In this manner we obtain as many equations as there are statically indeterminate reactions.

It can be shown that eqs. (200) represent the conditions for a minimum of function U, from which it follows that the magnitudes of statically indeterminate reactive forces are such as to make the strain energy of the system a minimum. This is the *principle of least work* as applied to the determination of redundant reactions.[23]

[23] The principle of least work was stated first by F. Menabrea in his article, "Nouveau principe sur la distribution des tensions dans les systèmes élastiques," Paris, C. R., Vol. 46 (1858), p. 1056. See also C.R., Vol. 98 (1884), p. 714. The complete proof of the principle was given by Castigliano, who made of this principle the fundamental method of solution of statically indeterminate systems. The application of strain energy methods in engineering was developed by O. Mohr (see his "Abhandlungen aus dem Gebiete d. technischen Mechanik"), by H. Müller-Breslau in his book, "Die neueren Methoden der Festigkeitslehre," and F. Engesser, "Über die Berechnung statisch unbestimmter Systeme," Zentralbl. d. Bauverwalt. 1907, p. 606. A very complete bibliography of this subject is given in the art. by M. Grüning, Encyklopädie d. Math. Wiss., Vol. IV, 2, II, p. 419.

As an example of application of the above principle let us consider a uniformly loaded beam built in at one end and supported at the other (Fig. 266). This is the problem with one statically indeterminate reaction. Taking the vertical reaction X at the right support as the statically indeterminate quantity, this unknown force is found from the equation:

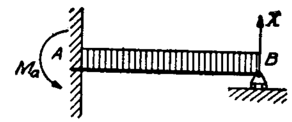

Fig. 266.

$$\frac{dU}{dX} = 0. \tag{a}$$

The strain energy of the beam, from eq. (187), is

$$U = \int_0^l \frac{M^2dx}{2EI}, \tag{b}$$

in which

$$M = Xx - \frac{qx^2}{2}.$$

Substituting in (*a*), we obtain

$$\frac{dU}{dX} = \frac{1}{EI}\int_0^l M\frac{dM}{dX}dx = \frac{1}{EI}\int_0^l\left(Xx - \frac{qx^2}{2}\right)xdx$$
$$= \frac{1}{EI}\left(X\frac{l^3}{3} - q\frac{l^4}{8}\right) = 0,$$

from which

$$X = \tfrac{3}{8}ql.$$

Instead of the reactive force X the reactive couple M_a at the left end of the beam could have been taken as the statically indeterminate quantity. The strain energy will now be a function of M_a. Equation (*b*) still holds, where now the bending moment at any cross section is

$$M = \left(\frac{ql}{2} - \frac{M_a}{l}\right)x - \frac{qx^2}{2}.$$

From the condition that the left end of the actual beam does not rotate when the beam is bent the derivative of the strain energy with respect to M_a must be equal to zero. From this we obtain

$$\frac{dU}{dM_a} = \frac{1}{EI}\int_0^l M\frac{dM}{dM_a}dx = -\frac{1}{EI}\int_0^l \left[\left(\frac{ql}{2} - \frac{M_a}{l}\right)x - \frac{qx^2}{2}\right]\frac{x}{l}dx = -\frac{1}{EI}\left(\frac{ql^3}{24} - \frac{M_al}{3}\right) = 0,$$

from which the absolute value of the moment is

$$M_a = \tfrac{1}{8}ql^2.$$

Problems in which we consider the forces acting in redundant members of the system as the statically indeterminate quantities can also be solved by using the Castigliano theorem. Take, as an example, the system represented in Fig. 15 which was already discussed (see p. 19). Considering the force X in the vertical bar OC as the statically indeterminated quantity, the forces in the inclined bars OB and OD are $(P - X)/2\cos\alpha$. Denoting by U_1 the strain energy of the inclined bars (Fig. 267, *a*) and by U_2 the strain energy of the vertical bar (Fig. 267, *b*), the total strain energy of the system is,[24]

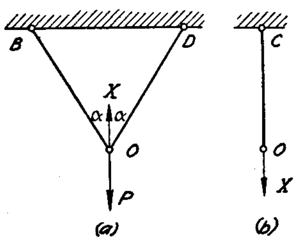

FIG. 267.

$$U = U_1 + U_2 = \left(\frac{P - X}{2\cos\alpha}\right)^2 \frac{l}{AE\cos\alpha} + \frac{X^2l}{2AE}. \qquad (c)$$

[24] It is assumed that all bars have the same cross-sectional area A and the same modulus of elasticity E.

If δ is the actual displacement downwards of the joint O in Fig. 15 the derivative with respect to X of the energy U_1 of the system in Fig. 267 (*a*) should be equal to $-\delta$, since the force X of the system has the direction opposite to that of the displacement δ. In the same time the derivative $\partial U_2/\partial X$ will be equal to δ. Hence

$$\frac{\partial U}{\partial X} = \frac{\partial U_1}{\partial X} + \frac{\partial U_2}{\partial X} = -\delta + \delta = 0. \qquad (d)$$

It is seen that the true value of the force X in the redundant member is such as to make the total strain energy of the system a minimum. Substituting for U its expression (*c*) in equation (*d*) we obtain

$$-\frac{(P - X)}{2\cos^2\alpha}\frac{l}{AE\cos\alpha} + \frac{Xl}{AE} = 0$$

from which

$$X = \frac{P}{1 + 2\cos^3\alpha}.$$

A similar reasoning can be applied to any statically indetermined system with one redundant member, and we can state that the force in that member is such as to make the strain energy of the system a minimum. To illustrate the procedure of calculating stresses in such systems let us consider the frame

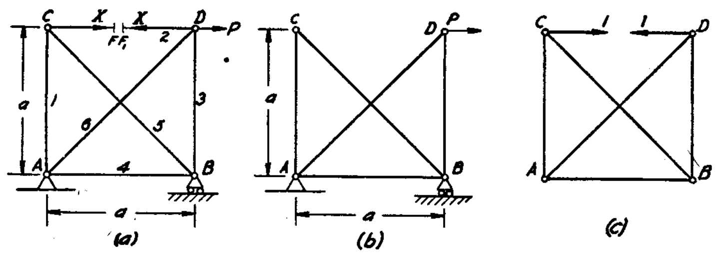

FIG. 268.

shown in Fig. 268 (*a*). The reactions here are statically determinate, but when we try to compute the forces in the bars, we find that there is one redundant member. Let us consider the bar CD as this redundant member. Cut this bar at any point and apply to each end F and F_1 a force X, equal to that in the bar. We thus arrive at a statically determinate system acted upon by the known force P, and, in addition, unknown forces X. The forces in the bars of this system are found in two groups: first, those produced by the external loading P assuming $X = 0$, Fig. 268, *b*, and denoted by S_i^0, where i indicates the number of the bar; second, those produced when the external force P is removed and unit forces replace the X forces (Fig. 268, *c*). The latter forces are denoted by S_i'. Then the total force in any bar, when the force P and the forces X are all acting, is

$$S_i = S_i^0 + S_i'X. \qquad (e)$$

The total strain energy of the system, from eq. (196), is

$$U = \sum_{i=1}^{i=m} \frac{S_i^2 l_i}{2A_iE} = \sum_{i=1}^{i=m} \frac{(S_i^0 + S_i'X)^2 l_i}{2A_iE}, \qquad (f)$$

in which the summation is extended over all the bars of the system including the bar CD, which is cut.[25] The Castigliano theorem is now applied and the derivative of U with respect to X gives the displacement of the ends F and F_1 towards each other. In the actual case the bar is continuous and this displacement is equal to zero. Hence

$$\frac{dU}{dX} = 0, \qquad (g)$$

i.e., the force X in the redundant bar is such as to make the strain energy of the system a minimum. From eqs. (*f*) and (*g*)

$$\frac{d}{dX}\sum_{i=1}^{i=m} \frac{(S_i^0 + S_i'X)^2 l_i}{2A_iE} = \sum_{i=1}^{i=m} \frac{(S_i^0 + S_i'X) l_i S_i'}{A_iE} = 0,$$

from which

$$X = -\frac{\displaystyle\sum_{i=1}^{i=m} \frac{S_i^0 S_i' l_i}{A_iE}}{\displaystyle\sum_{i=1}^{i=m} \frac{S_i'^2 l_i}{A_iE}}. \qquad (201)$$

[25] For this bar $S_i^0 = 0$ and $S_i' = 1$.

This process may be extended to a system in which there are several redundant bars.

The principle of least work can be applied also when the statically unknown quantities are couples. Take, as an example, a uniformly loaded beam on three supports (Fig. 269). If the bending moment at the middle support be considered the statically indeterminate quantity, the beam is cut at B and we obtain two simply supported beams (Fig.

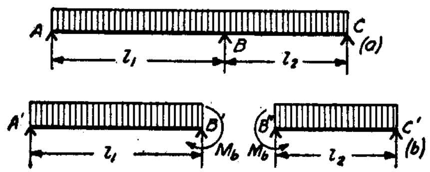

FIG. 269.

269, *b*) carrying the unknown couples M_b in addition to the known uniform load q. There is no rotation of the end B' with respect to the end B'' because in the actual case (Fig. 269, *a*) there is a continuous deflection curve. Hence

$$\frac{dU}{dM_b} = 0. \qquad (202)$$

Again the magnitude of the statically indeterminate quantity is such as to make the strain energy of the system a minimum.

Problems

1. The vertical load P is supported by a vertical bar DB of length l and cross-sectional area A and by two equal inclined bars of length l and cross sectional area A_1 (Fig. 270). Determine the forces in the bars and also the ratio A_1/A which will make the forces in all bars numerically equal.

Solution. The system is statically indeterminate. Let X be the tensile force in the vertical bar. The compressive forces in the inclined bars are $1/\sqrt{2}(P - X)$ and the strain energy of the system is

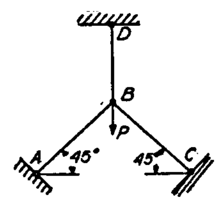

FIG. 270.

$$U = \frac{X^2 l}{2AE} + \frac{(P - X)^2 l}{2A_1E}.$$

The principle of least work gives

$$\frac{dU}{dX} = \frac{Xl}{AE} - \frac{(P - X)l}{A_1E} = 0,$$

from which

$$X = \frac{P}{1 + \frac{A_1}{A}}.$$

Substituting this into equation

$$X = \frac{1}{\sqrt{2}}(P - X),$$

we obtain

$$A_1 = \sqrt{2}A.$$

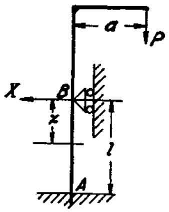

FIG. 271.

2. Determine the horizontal reaction X in the system shown in Fig. 271.

Solution. The unknown force X will enter only into the expression for the potential energy of bending of the portion AB of the bar. For this portion, $M = Pa - Xx$, and the equation of least work gives

$$\frac{dU}{dX} = \frac{d}{dX}\int_0^l \frac{M^2dx}{2EI} = \frac{1}{EI}\int_0^l M\frac{dM}{dX}dx = -\frac{1}{EI}\int_0^l (Pa - Xx)xdx$$
$$= \frac{1}{EI}\left(\frac{Xl^3}{3} - \frac{Pal^2}{2}\right) = 0,$$

from which

$$X = \frac{3}{2}P\frac{a}{l}.$$

3. Determine the horizontal reactions X of the system shown in Fig. 272. All dimensions are given in the table below.

Solution. From the principle of least work we have

$$\frac{dU}{dX} = \frac{d}{dX}\sum \frac{S_i^2l_i}{2A_iE}$$
$$= \sum \frac{S_il_i}{A_iE}\frac{dS_i}{dX} = 0.$$

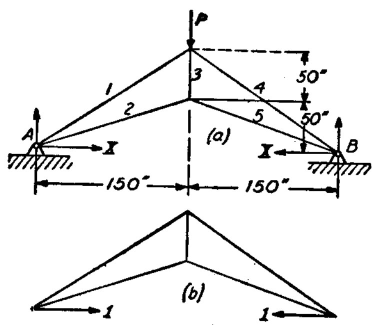

FIG. 272.

Let S_i^0 be the force in bar i produced by the known load P assuming $X = 0$, S_i' the force produced in the same bar by unit forces which replace the X forces (Fig. 272, *b*). The values of S_i^0 and S_i' are determined from statics. They are given in columns 4 and 5 of the table below. Then the total force in any bar is

$$S_i = S_i^0 + S_i'X.$$

i	l_i in.	A_i in.^{2}	S_i^0	S_i'	$\frac{S_i^0S_i'l_i}{A_i}$	$\frac{S_i'^2l_i}{A_i}$
1	180.3	5	$-1.803P$	1.202	$-78.1P$	52.0
2	158.1	3	$1.581P$	-2.108	$-175.7P$	234
3	50.0	2	$1.000P$	-1.333	$-33.3P$	44.5
4	180.3	5	$-1.803P$	1.202	$-78.1P$	52.0
5	158.1	3	$1.581P$	-2.108	$-175.7P$	234

$\Sigma = -540.9P$; $\Sigma = 616.5$.

Substituting into the equation of least work (200),

$$\sum_1^5 \frac{(S_i^0 + S_i'X)l_i}{A_iE}S_i' = 0,$$

from which

$$X = -\frac{\sum_{i=1}^{i=5}\frac{S_i^0S_i'l_i}{A_i}}{\sum_{i=1}^{i=5}\frac{S_i'^2l_i}{A_i}}. \qquad (f)$$

The necessary figures for calculating X are given in columns 6 and 7. Substituting this data into eq. (*f*), we obtain

$$X = 0.877P.$$

4. Determine the force in the redundant horizontal bar of the system shown in Fig. 273, assuming that the length of this bar is $l_0 = 300''$ and the cross-sectional area is A_0. The other bars have the same dimensions as in problem 3.

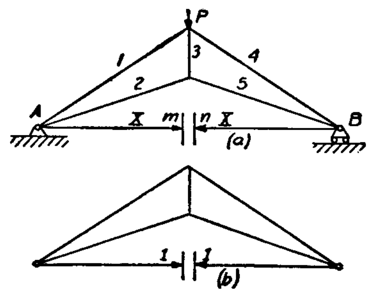

FIG. 273.

Solution. The force in the horizontal bar is calculated from eq. (201). This equation is of the same kind as eq. (f) in problem 3 but in the system of Fig. 273 there is the additional horizontal bar. The force produced in this bar by the force P alone ($X = 0$) is zero, i.e., $S_0^0 = 0$. The force produced by two forces equal to unity (Fig. 273, b) is $S_0' = 1$. The additional term in the numerator of eq. (f) is

$$\frac{S_0^0 S_0' l_0}{A_0} = 0.$$

The additional term in the denominator is

$$\frac{S_0'^2 l_0}{A_0} = \frac{1 \cdot l_0}{A_0} = \frac{300}{A_0}.$$

Then, by using the data of problem 3,

$$X = \frac{540.9P}{\frac{300}{A_0} + 616.5}.$$

Taking, for instance, $A_0 = 10$ sq. in.,

$$X = \frac{540.9P}{30 + 616.5} = 0.836P.$$

That is only 4.7 per cent less than the value obtained in problem 3 for immovable supports.[26]

Taking the cross-sectional area $A_0 = 1$ sq. in.,

$$X = \frac{540.9}{300 + 616.5} = 0.590P.$$

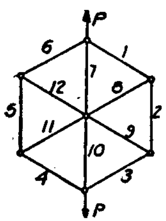

Fig. 274.

It can be seen that in statically indeterminate systems the forces in the bars depend also on their cross-sectional areas.

5. Determine the forces in the bars of the systems shown in Fig. 20 by using the principle of least work.

6. Determine the forces in the bars of the system shown in Fig. 274, assuming that all bars are of the same dimensions and material.

Solution. If one bar be removed, the forces in the remaining bars can be determined from statics; hence the system has one redundant bar. Let 1 be this bar and X the force acting in it. Then all the

[26] Taking $A_0 = \infty$, we obtain the same condition as for immovable supports.

bars on the sides of the hexagon will have tensile forces X, bars 8, 9, 11 and 12 have compressive forces X, and bars 7 and 10 have the force $P - X$. The strain energy of the system is

$$U = 10\frac{X^2 l}{2AE} + 2\frac{(P - X)^2 l}{2AE}.$$

From the equation $dU/dX = 0$ we obtain

$$X = \frac{P}{6}.$$

7. Determine the forces in the system shown in Fig. 268, assuming the cross-sectional areas of all bars equal and taking the force X in the diagonal AD as the statically indeterminate quantity.

Solution. Substituting the data, given in the table below, in eq. (201),

$$X = \frac{3 + 2\sqrt{2}}{4 + 2\sqrt{2}}P.$$

i	l_i	S_i^0	S_i'	$S_i^0 S_i' l_i$	$S_i'^2 l_i$
1	a	P	$-1/\sqrt{2}$	$-aP/\sqrt{2}$	$a/2$
2	a	P	$-1/\sqrt{2}$	$-aP/\sqrt{2}$	$a/2$
3	a	0	$-1/\sqrt{2}$	0	$a/2$
4	a	P	$-1/\sqrt{2}$	$-aP/\sqrt{2}$	$a/2$
5	$a\sqrt{2}$	$-P\sqrt{2}$	$+1$	$-2aP$	$a\sqrt{2}$
6	$a\sqrt{2}$	0	$+1$	0	$a\sqrt{2}$

$$\Sigma = \frac{-(3 + 2\sqrt{2})aP}{\sqrt{2}}; \quad \Sigma = 2a(1 + \sqrt{2}).$$

8. A rectangular frame of uniform cross section (Fig. 275) is submitted to a uniformly distributed load of intensity q as shown. Determine the bending moment M at the corner.

Answer.

$$M = \frac{(a^3 + b^3)q}{12(a + b)}.$$

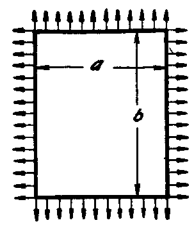

Fig. 275.

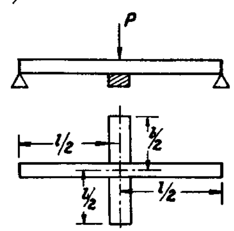

Fig. 276.

9. A load P is supported by two beams of equal cross section, crossing each other as shown in Fig. 276. Determine the pressure X between the beams.

Answer.

$$X = \frac{Pl^3}{l^3 + l_1^3}.$$

10. Find the statically indeterminate quantity in the frame shown in Fig. 167 by using the principle of least work.

Solution. The strain energy of bending of the frame is

$$U = 2\int_0^h \frac{H^2x^2dx}{2EI_1} + \int_0^l \frac{(M_0 - Hh)^2dx}{2EI}, \tag{g}$$

in which M_0 denotes the bending moment for the horizontal bar calculated as for a beam simply supported at the ends. Substituting in equation:

$$\frac{dU}{dH} = 0, \tag{h}$$

we find

$$\frac{2H}{EI_1}\frac{h^3}{3} + \frac{Hh^2l}{EI} = \frac{h}{EI}\int_0^l M_0dx. \tag{k}$$

The integral on the right side is the area of the triangular moment diagram for a beam carrying the load P. Hence

$$\int_0^l M_0dx = \frac{1}{2}Pc(l - c).$$

Substituting in (k), we obtain for H the same expression as in (114). (See p. 192.)

11. Find the statically indeterminate quantities in the frames shown in Figs. 166, 169 and 171 by using the principle of least work.

12. Find the bending moment in Fig. 269 assuming that $l_1 = 2l_2$.

71. The Reciprocal Theorem.—Let us begin with a problem of a simply supported beam shown in Fig. 277 (a) and calculate the deflection at a point D when the load P is acting at C. This deflection is obtained by substituting $x = d$ into equation (86) which gives

$$(y)_{x=d} = \frac{Pbd}{6l}(l^2 - b^2 - d^2). \tag{a}$$

It is seen that the deflection (a) does not change if we put d for b and b for d, which indicates that for the case shown in Fig. 277 (b) the deflection at D_1 is the same as the deflection at D in Fig. 277 (a). From Fig. 277 (b) we obtain Fig. 277 (c) by simply rotating the beam through 180 degrees which brings point C_1 in coincidence with point D and point D_1 with point C. Hence the deflection at C in Fig. 277 (c) is equal to the deflection at D in Fig. 277 (a). This means that if the load P is moved from point C to point D, the deflection which in the first case of loading was measured at D will be obtained in the second case at point C. This is a particular case of the *reciprocal theorem.*

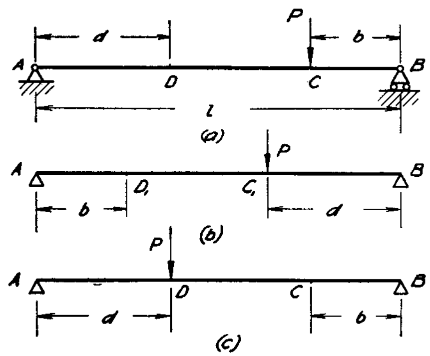

Fig. 277.

To establish the theorem in general form [27] we consider an elastic body, shown in Fig. 278, loaded in two different manners and supported in such a way that displacement as a rigid body is impossible. In the first state of stress the applied forces are P_1 and P_2, and in the second state P_3 and P_4. The displacements of the points of application in the directions of the forces are $\delta_1, \delta_2, \delta_3, \delta_4$ in the first state and $\delta_1', \delta_2', \delta_3', \delta_4'$ in the second state. The reciprocal theorem states: The work done by the forces of the first state on the corresponding displacements of the second state is equal to the work done by the forces of the second state on the corresponding dis-

[27] A particular case of this theorem was obtained by J. C. Maxwell, loc. cit., p. 317. The theorem is due to E. Betti, Il nuovo Cimento (Ser. 2), V. 7 and 8 (1872). In a more general form, the theorem was given by Lord Rayleigh, London Math. Soc. Proc., Vol. 4 (1873), or Scientific Papers, Vol. 1, p. 179. Various applications of this theorem to the solution of engineering problems were made by O. Mohr, loc. cit., p. 327, and H. Müller-Breslau, loc. cit., p. 327.

placements of the first. In symbols this means

$$P_1\delta_1' + P_2\delta_2' = P_3\delta_3 + P_4\delta_4. \qquad (203)$$

To prove this theorem let us consider the strain energy of the body when all forces $P_1, \cdots P_4$ are acting together and let us

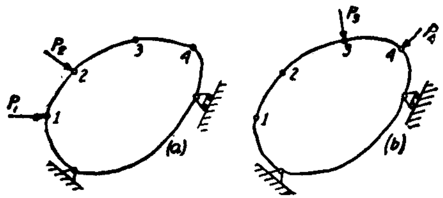

FIG. 278.

use the fact that the amount of the strain energy does not depend upon the order in which the forces are applied but only upon the final values of the forces. In the first manner of loading assume that forces P_1 and P_2 are applied first and later forces P_3 and P_4. The strain energy stored during the application of P_1 and P_2 is

$$\frac{P_1\delta_1}{2} + \frac{P_2\delta_2}{2}. \qquad (a)$$

Applying now P_3 and P_4, the work done by these forces is

$$\frac{P_3\delta_3'}{2} + \frac{P_4\delta_4'}{2}. \qquad (b)$$

It must be noted, however, that during the application of P_3 and P_4 the points of application of the previously applied forces P_1 and P_2 will be displaced by δ_1' and δ_2'. Then P_1 and P_2 do the work

$$P_1\delta_1' + P_2\delta_2'.\text{[28]} \qquad (c)$$

Hence the total strain energy stored in the body, by summing

[28] These expressions are not divided by 2 because forces P_1 and P_2 remain constant during the time in which their points of application undergo the displacements δ_1' and δ_2'.

(a), (b) and (c), is

$$U = \frac{P_1\delta_1}{2} + \frac{P_2\delta_2}{2} + \frac{P_3\delta_3'}{2} + \frac{P_4\delta_4'}{2} + P_1\delta_1' + P_2\delta_2'. \qquad (d)$$

In the second manner of loading, let us apply the forces P_3 and P_4 first and afterwards P_1 and P_2. Then, repeating the same reasoning as above, we obtain

$$U = \frac{P_3\delta_3'}{2} + \frac{P_4\delta_4'}{2} + \frac{P_1\delta_1}{2} + \frac{P_2\delta_2}{2} + P_3\delta_3 + P_4\delta_4. \qquad (e)$$

Putting (d) and (e) equal, eq. (203) is obtained. This theorem can be proven for any number of forces, and also for couples, or for forces and couples. In the case of a couple the corresponding angle of rotation is considered as the displacement.

For the particular case in which a single force P_1 acts in the first state of stress, and a single force P_2 in the second state, eq. (203) becomes [29]

$$P_1\delta_1' = P_2\delta_2. \qquad (204)$$

If $P_1 = P_2$, it follows that $\delta_1' = \delta_2$, i.e., the displacement of the point of application of the force P_2 in the direction of this force, produced by the force P_1, is equal to the displacement of the point of application of the force P_1 in the direction of P_1, produced by the force P_2. A verification of this conclusion for a particular case was given in considering the beam shown in Fig. 277.

As another example let us again consider the bending of a simply supported beam. In the first state let it be bent by a load P at the middle, and in the second state by a bending couple M at the end. The load P produces the slope $\theta = Pl^2/16EI$ at the end. The couple M, applied at the end, produces the deflection $Ml^2/16EI$ at the middle. Equation (204) becomes

$$P\frac{Ml^2}{16EI} = M\frac{Pl^2}{16EI}.$$

[29] This was proved first by J. C. Maxwell, and is frequently called Maxwell's theorem.

The reciprocal theorem is very useful in the problem of finding the most unfavorable position of moving loads on a statically indeterminate system. An example is shown in Fig. 279, which represents a beam built in at one end and simply supported at the other and carrying a concentrated load P. The problem is to find the variation in the magnitude of the reaction X at the left support as the distance x of the load from this support changes.

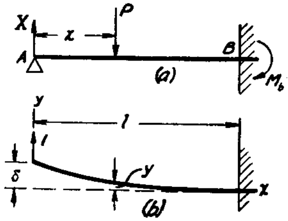

FIG. 279.

Let us consider the *actual condition* of the beam (Fig. 279, *a*) as the first state of stress. The second, or *fictitious*, state is shown in Fig. 279 (*b*). The external load and the redundant support are there removed and a unit force upward replaces the unknown reaction X. This second state of stress is statically determinate and the corresponding deflection curve is known (see eq. 97, p. 148). If the coordinate axes are taken as shown in Fig. 279 (*b*),

$$y = \frac{1}{6EI}(l - x)^2(2l + x). \qquad (f)$$

Let δ denote the deflection at the end and y the deflection at distance x from the left support. Then, applying the reciprocal theorem, the work done by the forces of the first state on the corresponding displacements of the second state is

$$X\delta - Py.$$

In calculating now the work done by the forces of the second state, there is only the unit force on the end,[30] and the corresponding displacement of the point A in the first state is equal to zero. Consequently this work is zero and the reciprocal theorem gives

$$X\delta - Py = 0,$$

from which

$$X = P\frac{y}{\delta}. \qquad (g)$$

[30] The reactions at the built-in end are not considered in either case because the corresponding displacement is zero.

It is seen that, as the load P changes position, the reaction X is proportional to the corresponding values of y in Fig. 279 (*b*). Hence the deflection curve of the second state (eq. *f*) gives a complete picture of the manner in which X varies with x. Such a curve is called the *influence line* for the reaction X.[31]

If several loads act simultaneously, the use of eq. (*g*) together with the method of superposition gives

$$X = \frac{1}{\delta}\sum P_n y_n,$$

where y_n is the deflection corresponding to the load P_n and the summation is extended over all the loads.

Problems

1. Construct the influence lines for the reactions at the supports of the beam on three supports (Fig. 280).

Solution. To get the influence line for the middle support the actual state shown in Fig. 280 (*a*) is taken as the first state of stress. The second state is indicated in Fig. 280 (*b*), in which the load P is removed and the reaction X is replaced by a unit force upward. This second state of stress is statically determinate and the deflection curve is known (eqs. 86 and 87, p. 142); hence the deflections δ and y can be calculated. Then the work done by the forces of the first state on the corresponding displacements of the second state is

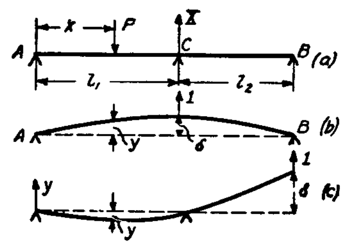

FIG. 280.

$$X\delta - Py.$$

The work of the forces of the second state (force-unity) on the corresponding displacements of the first state (zero deflection at C) is zero; hence

$$X\delta - Py = 0; \qquad X = P\frac{y}{\delta}.$$

Hence the deflection curve of the second state is the influence line for the reaction X. In order to get the influence line for the reac-

[31] The use of models in determining the influence lines was developed by G. E. Beggs, Journal of Franklin Institute 1927.

tion at B, the second state of stress should be taken as shown in Fig. 280 (c).

2. By using the influence line of the previous problem, determine the reaction at B if the load P is at the middle of the first span ($x = l_1/2$) (Fig. 280, a).

Answer. Reaction is downward and equal to

$$\frac{3P}{16}\frac{l_1^2}{l_2^2 + l_2 l_1}.$$

3. Find the influence line for the bending moment at the middle support C of the beam on three supports (Fig. 281). By using this line calculate the bending moment M_c when the load P is at the middle of the second span.

Solution. The first state of stress is the actual state (Fig. 281, a) with a bending moment M_c acting at the cross section C. For the second state of stress the load P is removed, the beam is cut at C and two equal and opposite unit couples replace M_c (Fig. 281, b). This case is statically determinate. The angles θ_1 and θ_2 are given by eq. (104) and the deflection y by eq. (105). The sum of the angles $\theta_1 + \theta_2$ represents the displacement in the second state corresponding to the bending moment M_c acting in the first state.

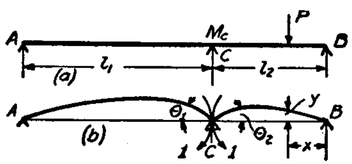

FIG. 281.

The work done by the forces of the first state on the corresponding displacement in the second state is [32]

$$M_c(\theta_1 + \theta_2) - Py.$$

The work done by the forces of the second state on the displacements of the first state is zero because there is no cut at the support C in the actual case and the displacement corresponding to the two unit couples of the second state is zero. Hence

$$M_c(\theta_1 + \theta_2) - Py = 0$$

and

$$M_c = P\frac{y}{\theta_1 + \theta_2}. \qquad (h)$$

It will be seen that as the load P changes its position, the bending moment M_c changes in the same ratio as the deflection y. Hence

[32] It is assumed that the bending moment M_c produces a deflection curve concave downward.

the deflection curves of the second state represent the influence line for M_c. Noting that

$$\theta_1 + \theta_2 = \frac{l_1 + l_2}{3EI}$$

and that the deflection at the middle of the second span is

$$(y)_{x=l_2/2} = \frac{1 \cdot l_2^2}{16EI},$$

the bending moment when the load P is at the middle of the second span is, from eq. (h),

$$M_c = \frac{3}{16}\cdot\frac{Pl_2^2}{l_1 + l_2}.$$

The positive sign obtained for M_c indicates that the moment has the direction indicated in Fig. 281 (b). Following our general rule for the sign of moments (Fig. 58) we then consider M_c as a negative bending moment.

4. Find the influence line for the bending moment at the built-in end B of the beam AB shown in Fig. 279, and calculate this moment when the load is at the distance $x = l/3$ from the left support.

Answer.

$$M_b = (4/27)lP.$$

5. Construct the influence line for the horizontal reactions H of the frame shown in Fig. 167 (a) as the load P moves along the bar AB.

Answer. The influence line has the same shape as the deflection curve of the bar AB for the loading condition shown in Fig. 166 (c).

6. Construct the influence line for the force X in the horizontal bar CD (Fig. 282, a) as the load P moves along the beam AB. Calculate X when the load is at the middle. The displacements due to elongation and contraction of the bars are to be neglected and only the displacement due to the bending of the beam AB is to be taken into account.

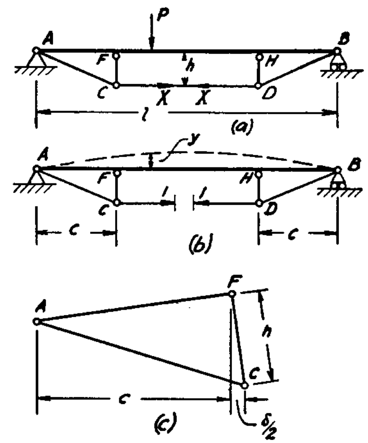

FIG. 282.

Solution. The actual condition (Fig. 282, *a*) is taken as the first state of stress. In the second state the load P is removed and the forces X are replaced by unit forces (Fig. 282, *b*). Due to these forces, upward vertical pressures equal to $(1 \cdot h)/c$ are transmitted to the beam AB at the points F and H and the beam deflects as indicated by the dotted line. If y is the deflection of the beam at the point corresponding to the load P, and δ is the displacement of the points C and D towards one another in the second state of stress, the reciprocal theorem gives

$$X\delta - Py = 0 \quad \text{and} \quad X = P\frac{y}{\delta}. \tag{i}$$

Hence the deflection curve of the beam AB in the second state is the required influence line. The bending of the beam by the two symmetrically situated loads is discussed in problem 1, p. 159. Substituting $(1 \cdot h)/c$ for P in the formulas obtained there, the deflection of the beam at F and that at the middle are

$$(y)_{x=c} = \frac{ch}{6EI}(3l - 4c) \quad \text{and} \quad (y)_{x=l/2} = \frac{h}{24EI}(3l^2 - 4c^2),$$

respectively.

Considering the rotation of the triangle AFC (Fig. 282, *c*) as a rigid body, the horizontal displacement of the point C is equal to the vertical displacement of the point F multiplied by h/c; hence

$$\delta = 2\frac{h}{c}(y)_{x=c} = \frac{h^2}{3EI}(3l - 4c).$$

Substituting this and the deflection at the middle into eq. (*i*) gives

$$X = \frac{P}{8h}\,\frac{3l^2 - 4c^2}{3l - 4c}.$$

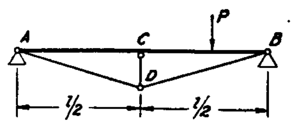

FIG. 283.

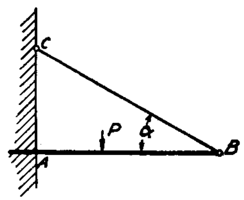

FIG. 284.

7. Find the influence line for the force in the bar CD of the system shown in Fig. 283, neglecting displacements due to contractions and elongations and considering only the bending of the beam AB.

Answer. The line will be the same as that for the middle reaction of the beam on three supports (see problem 1, p. 335).

8. Construct the influence line for the bar BC which supports the beam AB. Find the force in BC when P is at the middle (Fig. 284).

Answer. Neglecting displacements due to elongation of the bar BC and contraction of the beam AB, the force in BC is $\frac{5}{16}(P/\sin\alpha)$.

72. Exceptional Cases.—In the derivation of both the Castigliano theorem and the reciprocal theorem it was assumed that the displacements due to strain are proportional to the loads acting on the elastic system. There are cases in which the displacements are not proportional to the loads, although the material of the body may follow Hooke's law. This always occurs when the displacements due to deformations must be considered in discussing the action of external loads. In such cases, the strain energy is no longer a second degree function and the theorem of Castigliano does not hold. In order to explain this limitation let us consider a simple case in which only one force P acts on the elastic system. Assume first that the displacement δ is proportional to the corresponding force P as represented by the straight line OA in Fig. 285

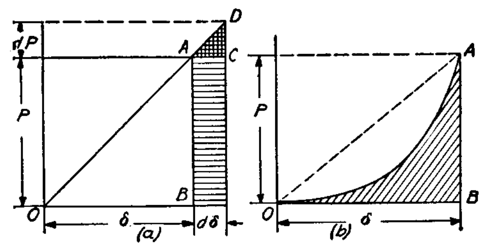

FIG. 285.

(*a*). Then the area of the triangle OAB represents the strain energy stored in the system during the application of the load P. For an infinitesimal increase $d\delta$ in the displacement the strain energy increases by an amount shown in the figure by the shaded area and we obtain

$$dU = Pd\delta. \tag{a}$$

With a linear relationship the infinitesimal triangle ADC is similar to the triangle OAB; therefore

$$\frac{d\delta}{dP} = \frac{\delta}{P} \quad \text{or} \quad d\delta = \frac{dP\delta}{P}. \tag{b}$$

Substituting this into eq. (*a*),

$$dU = P\frac{dP\delta}{P},$$

from which the Castigliano statement is obtained:

$$\frac{dU}{dP} = \delta. \qquad (c)$$

An example to which the Castigliano theorem cannot be applied is shown in Fig. 286. Two equal horizontal bars AC and BC hinged at A, B and C are subjected to the action of the vertical force P at C. Let C_1 be the position of C after deformation and α the angle of inclination of either bar in its deformed condition. The unit elongation of the bars, from Fig. 286 (*a*), is

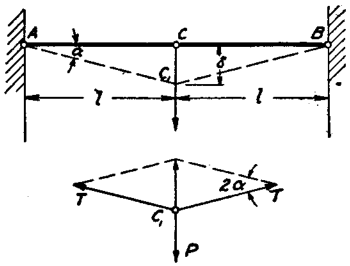

FIG. 286.

$$\epsilon = \left(\frac{l}{\cos\alpha} - l\right) : l. \qquad (d)$$

If only small displacements are considered, α is small and $1/\cos\alpha = 1 + (\alpha^2/2)$ approximately. Then, from (*d*),

$$\epsilon = \frac{\alpha^2}{2}.$$

The corresponding forces in the bars are

$$T = AE\epsilon = \frac{AE\alpha^2}{2}. \qquad (e)$$

From the condition of equilibrium of the point C_1 (Fig. 286, *b*),

$$P = 2\alpha T, \qquad (f)$$

and for T, as given in eq. (*e*),

$$P = AE\alpha^3,$$

from which

$$\alpha = \sqrt[3]{\frac{P}{AE}} \qquad (g)$$

and

$$\delta = l\alpha = l\sqrt[3]{\frac{P}{AE}}. \qquad (205)$$

In this case the displacement is not proportional to the load P, although the material of the bars follows Hooke's law. The relation between δ and P is represented in Fig. 285 (*b*) by the curve OA. The shaded area OAB in this figure represents the strain energy stored in the system. The amount of strain energy is

$$U = \int_0^\delta P d\delta. \qquad (h)$$

Substituting, from (205),

$$P = AE\frac{\delta^3}{l^3}, \qquad (i)$$

we obtain

$$U = \frac{AE}{l^3}\int_0^\delta \delta^3 d\delta = \frac{AE\delta^4}{4l^3} = \frac{P\delta}{4} = \frac{Pl}{4}\sqrt[3]{\frac{P}{AE}}. \qquad (l)$$

This shows that the strain energy is no longer a function of the second degree in the force P. Also it is not one half but only one quarter of the product $P\delta$ (see art. 68). The Castigliano theorem of course does not hold here:

$$\frac{dU}{dP} = \frac{d}{dP}\left(\frac{Pl}{4}\sqrt[3]{\frac{P}{AE}}\right) = \frac{1}{3}l\sqrt[3]{\frac{P}{AE}} = \frac{1}{3}\delta.$$

Analogous results are obtained in all cases in which the displacements are not proportional to the loads.