2.1. Introduction

For a simple tension or compression load, only the area of the cross-section is needed to study or verify the strength of a section of a beam, for example. For all other types of loading, the shape and dimensions of the beam's cross-section play a major role in its behavior under different types of stress, such as torsion   or   bending.

In this chapter, we will focus on the following characteristics:

- Area of a section
- Static moment
- Center of gravity
- moment of inertia (quadratic moment)

2.2. Area of a Section

By definition, the area A of a section is given by the integral:

$$A = \int_A dA \qquad\qquad 2.1$$

- **Example 2.1**

Calculate the area of a triangle.

- **Solution 2.1**

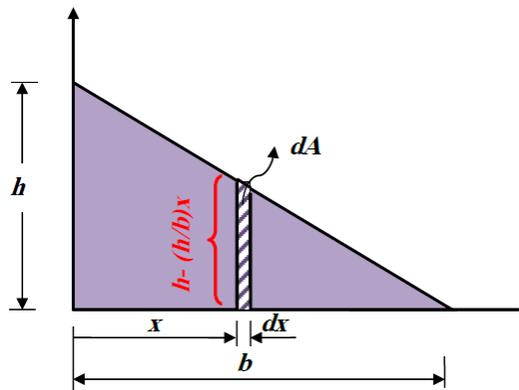Consider the planar triangular surface shown in the figure below.

Fig 2.1

Let us consider an elementary surface such that:

$$dA = h\left(1 - \frac{x}{b}\right)dx$$

$$A = \int_A dA = \int_0^b h\left(1 - \frac{x}{b}\right)dx = \frac{bh}{2}$$

Remark

If the section is composite, we divide it into standard (regular) sections, and the area is calculated as

$$A = \sum_{i=1}^{n} A_i$$

2.3. Static Moment

The static moment **S** of a section with respect to an axis ox or oy (Fig. 2.1) is given by one of the following expressions

$$S_x = \int_A y\,dA \qquad\qquad 2.2$$
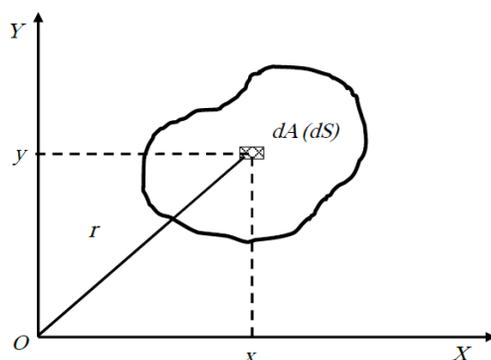
$$S_y = \int_A x\,dA \qquad\qquad 2.3$$

Fig 2.2. Plane section.

If translations are made parallel to the ox and oy axes, the static moments change. Let the section shown in figure (2.2) be such that S_X, S_Y, and A are known, and we aim to determine $S_{X'}$ and $S_{Y'}$.
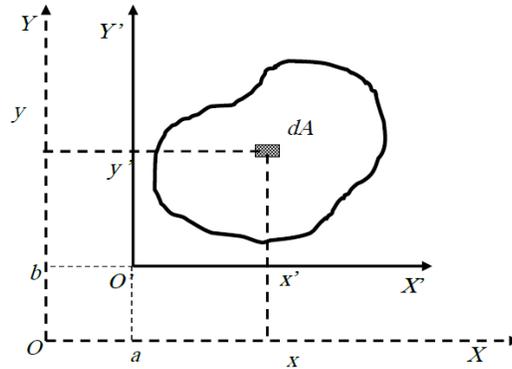
Fig 2.3. Translation of the axes.

From figure (2.3), we have:

$$x' = x - a \qquad\qquad\qquad y' = y - b$$

By definition, we have:

$$S_{X'} = \int_A y' \, dA = \int_A (y - b) \, dA$$

$$S_{Y'} = \int_A x' \, dA = \int_A (x - a) \, dA$$

Where:

$$S_{X'} = S_x - b.A \qquad\qquad\qquad 2.4$$
$$S_{Y'} = S_y - a.A \qquad\qquad\qquad 2.5$$

2.4. Center of gravity

We can choose a and b so that $S_{X'}$ and $S_{Y'}$ are zero, that is:

$$a = S_Y/A \qquad b = S_X/A$$

- The axis for which the static moment is zero is called the central axis

- The point of intersection of two central axes is called the center of gravity of a section.

Thus, the coordinates of the center of gravity of a section are written:

$$x_G = S_Y/A \qquad\qquad y_G = S_x/A \qquad\qquad 2.6$$

- **Definition**

The center of gravity (G) of a section is the point such that the static moment of the section with respect to any axis passing through this point is zero.

We can say that the static moment of a section is equal to the product of the area of the section and the distance between its center of gravity G and the axis.

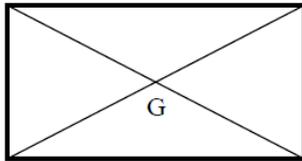Figures (2.4) and (2.5) show examples of positions of centers of gravity.

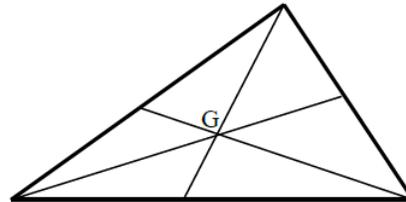**Fig 2.4.** Aire rectangulaire.   **Fig 2.5.** Triangular area.

- **Remark**

  For a composite section, the coordinates of the center of gravity are given by the following expressions:

$$S_x = \sum y_{G_i}.A_i; \quad i = 1, n \qquad\qquad 2.7$$

$$S_y = \sum x_{G_i}.A_i; \quad i = 1, n \qquad\qquad 2.8$$

- **Example 2.3**

Determine the coordinates of the center of gravity of the triangular section shown below.
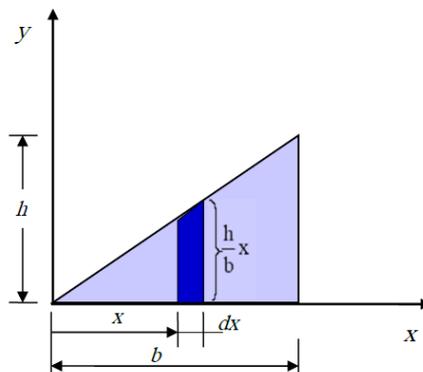

Fig 2.6

- **Solution 2.3**

$$X_G = \frac{\int_A x \, dA}{\int_A dA} = \frac{\int_0^b x \left(\frac{h}{b} x \, dx\right)}{\int_0^b \frac{h}{b} x \, dx}$$

Where:

$$X_G = \frac{2}{3} b$$

$$Y_G = \frac{\int_A y \, dA}{\int_A dA} = \frac{\int_0^b x \left(\frac{h}{b} x \, dx\right)}{\int_0^b \frac{h}{b} x \, dx}$$

Where:

$$Y_G = \frac{1}{3} h$$

- **Properties**

If the section has an axis of symmetry, the center of gravity G is located on this axis. If there are no axes of symmetry, proceed as follows:

- Choose a reference frame (O, x, y)
- Calculate the static moment S of the section with respect to the axes of the reference frame
- Calculate the total area of the section
- Use the static moment property $S_Y = X_G . A \quad S_X = Y_G . A$

- **Example 2.4**

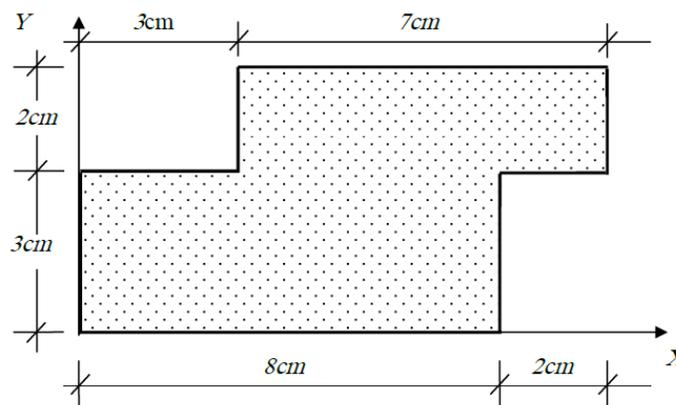Calculate the coordinates of the center of gravity of the following plane section

Fig 2.7.

- **Solution 2.4**

$S_X = 2.5(5 \times 10) - 4(2 \times 3) - 1.5(3 \times 2) = 125 - 24 - 9 = 92 \text{cm}^3$

$S_Y = 5(5 \times 10) - 1.5(2 \times 3) - 9(3 \times 2) = 250 - 9 - 54 = 187 \text{cm}^3$

$X_G = S_Y / A = 187/38 = 4.9 \text{cm}$

$Y_G = S_X / A = 92/38 = 2.4 \text{cm}$

2.5. Moment of Inertia

2.5.1. Definition

The moment of inertia, or quadratic moment, of a section is defined as the degree of resistance of this section to applied external forces, taking into account the shape of this section.

By definition, the integrals:

$$I_x = \int_A y^2 dA \tag{2.9}$$

$$I_y = \int_A x^2 dA \tag{2.10}$$

They are called the moments of inertia of section A with respect to the ox and oy axes, respectively, as shown in figure 2.8. These expressions are derived from the following definition.

The moment of inertia of an infinitely small surface with respect to an axis distant from that surface is equal to the product of its area and the square of its distance from the axis. It is always positive and is expressed in m⁴, cm⁴, or mm⁴.

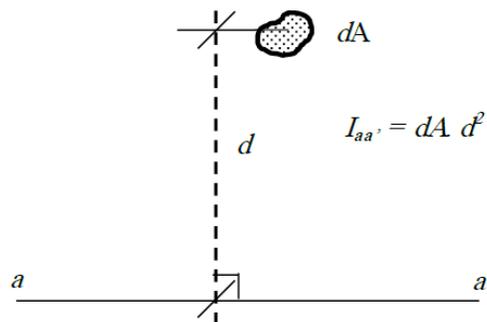**Fig 2.8.** Quadratic moment of a section.

The integral:

$$I_{xy} = \int_A xy \, dA \qquad\qquad 2.11$$

It is called the centrifugal moment or product of inertia of section A with respect to the xoy system.

- **Remark**

The quadratic moments I_x and I_y are always positive, while the product moment Ixy can be positive, negative, or zero.

- **Example 2.5**

Calculate the quadratic moments with respect to the o'x' and o'y' axes, as well as the produced moment for the rectangle shown in the following figure.
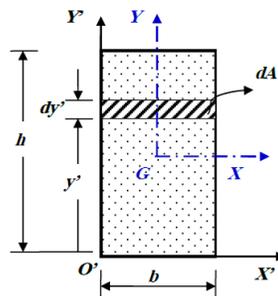

Fig 2.9

- **Solution 2.5**

$$I_{x'} = \int_A y'^2 \, dA$$

$$I_{x'} = \int_A^H y'^2 . b . dy' = \frac{bh^3}{3}$$

In the same way

$$\int_A x'^2 \, dA = \frac{b^3 h}{3}$$

And

$$I_{x'y'} = \int_A x' y'^2 \, dA$$

$$I_{x'y'} = \int_0^H \int_0^B x'y'dx'dy' = \frac{b^3 h}{3}$$

2.5.2. Polar moment of inertia

The polar moment of inertia of the section shown in Figure 4.1 is given by the relationship:

$$I_P = \int_A r^2 \, dA \qquad\qquad 2.12$$

With

$$r^2 = x^2 + y^2$$

Where

$$I_P = I_x + I_y \qquad\qquad 2.13$$

The polar moment of inertia is always positive and never zero.

- **Theorem**

The polar moment of inertia of a section with respect to any point of that section is equal to the sum of the moments of inertia with respect to two perpendicular axes passing through that point.

- **Example 2.6**

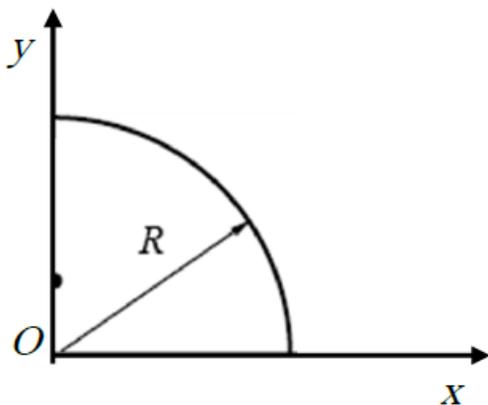For the quarter circle shown in figure (2.10, and 2.11), calculate the quadratic polar moment IO.
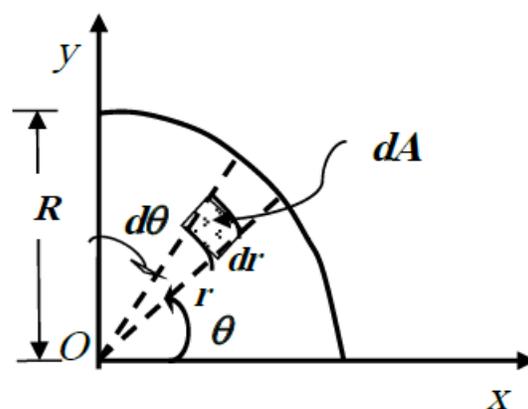

Fig 2.10.   **Fig 2.11.**

• **Solution 2.6**

From the definition of the polar moment of inertia and figure (2.10, and 2.11) we write:

$$I_O = \int_A r^2 dA = \int_A r^2 (r dr\, d\theta)$$

$$I_O = \left(\int_0^R r^3 dr \right) \left(\int_0^{\frac{\pi}{2}} d\theta \right) = \frac{\pi R^4}{8}$$

Or in terms of diameter

$$I_O = \frac{\pi D^4}{128}$$

2.6. Variations of Moments of Inertia
2.6.1. Translation of Axes

Let there be a section A, with its moments of inertia in the xoy coordinate system denoted as I_x, I_y, and I_{xy}. We aim to calculate the moments of inertia of section A in the x'o'y' coordinate system by translating the axes **ox** and **oy** as shown in Figure 2.12.

$$x' = x + a \;\;; \qquad y' = y + b$$

$$I_{x'} = \int_A y'^2 dA = \int_A (y + b)^2 dA$$

$$I_{x'} = \int_A y^2 dA + 2b \int_A y^2 dA + b^2 \int_A dA$$

Where

$$I_{x'} = I_x + 2bS_x + b^2 A \qquad\qquad 2.14$$

We follow the same reasoning for $I_{y'}$ and $I_{x'y'}$

If point O coincides with the center of gravity G, the static moments S_x and S_y become zero and we have:

$$I_{x'} = I_x + b^2 A \qquad\qquad 2.15$$
$$I_{y'} = I_y + a^2 A \qquad\qquad 2.16$$
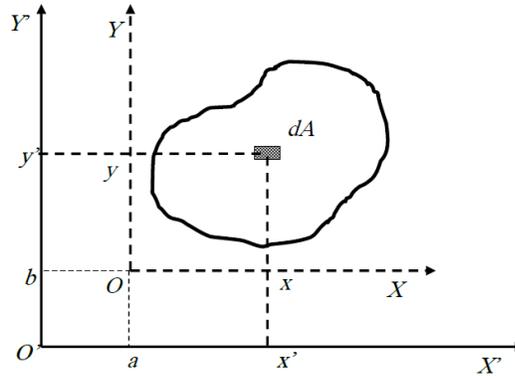$$I_{x'y'} = I_{xy} + abA \qquad\qquad 2.17$$

Fig 1.12. Moment of inertia of a section and translation of the axes.

- **Huygens' Theorem**

Let the axis Δ_G pass through the center of gravity G of the surface S. The moment of inertia (quadratic moment) I_Δ is calculated from the moment of inertia (quadratic moment) $I_{\Delta G}$ using the formula:

$$I_\Delta = I_{\Delta G} + d^2.A \qquad\qquad 2.18$$

Where d is the distance between the axis Δ and Δ_G.

Applying Huygens' theorem to the axes $\left(\overrightarrow{o,x'}\right)$, and $\left(\overrightarrow{o,y'}\right)$ gives:

$$I_x = I_{Gx'} + Y_G^2.A \qquad\qquad 2.19$$

$$I_y = I_{Gy'} + X_G^2.A \qquad\qquad 2.20$$

If a surface S is composed of n surfaces S_i, we have:

$$I_{Oxi} = \sum_1^n I_{Ox} \qquad \text{and} \qquad I_{Oyi} = \sum_1^n I_{Oy}$$

With:

- I_{Oxi}: quadratic moment of surface S_i relative to the axis$(0,\vec{x})$

- I_{Oyi}: quadratic moment of surface S_i relative to the axis $(0, \vec{y})$:
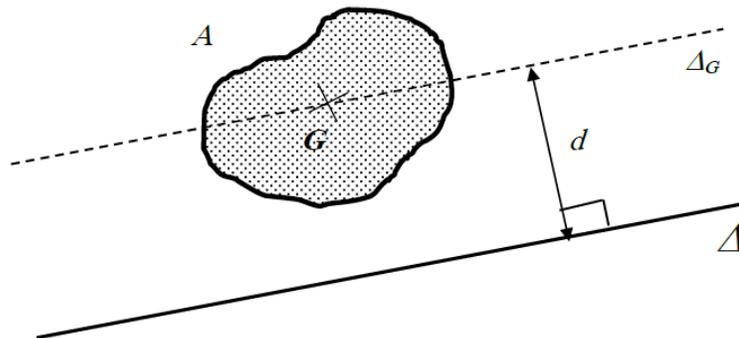

Fig 2.13. Schematization of Huygens' theorem.

- **Example 2.7**

Determine the moments of inertia with respect to the xoy system for the rectangle shown in the figure below.
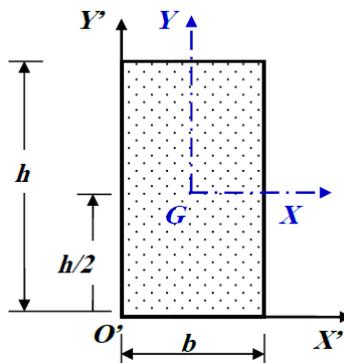

Fig 2.14.

- **Solution 2.7**

From Huygens's relation we write:

$$I_x = I_{x'} - d^2 A$$

$$I_x = \frac{bh^3}{3} - \left(\frac{h}{2}\right)^2 bh = \frac{bh^3}{12}$$

And

$$I_y = I_{y'} - d^2 A$$

$$I_y = \frac{b^2 h^2}{4} - \frac{b}{2}\frac{h}{2} bh = 0$$

Similarly

$$I_{xy} = I_{x'y'} - abA$$

$$I_{xy} = \frac{b^2 h^2}{4} - \frac{b}{2}\frac{h}{2}bh = 0$$

Because the x and y axes are central.

2.6.2. Rotation of Axes

Let a section A have known moments of inertia in the xoy coordinate system: I_x, I_y, and I_{xy}. We aim to calculate the moments of inertia of section A in the uov coordinate system, which is inclined at an angle θ to the xoy system (Fig. 2.15).
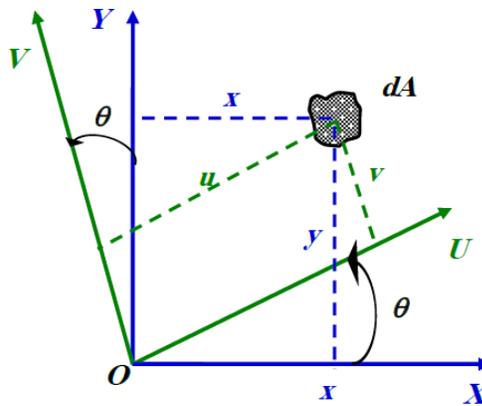

Fig 2.15. Moment of inertia of a section and rotation of the axes.

According to figure (2.15)

$$u = x \cos \theta + y \sin \theta$$
$$v = -x \sin \theta + y \cos \theta$$

Using the definition of the moment of inertia, we write:

$$I_u = \int_A v^2 \, dA$$

$$I_u = \cos^2 \theta \int_A y^2 dA + \sin^2 \theta \int_A x^2 dA - 2 \sin \theta \cos \theta \int_A xy dA$$

$$I_u = \cos^2 \theta . I_x + \sin^2 \theta . I_y - 2 \sin \theta \cos \theta . I_{xy} \qquad\qquad 2.21$$

Using trigonometric relationships

$$\sin^2 \theta = \frac{1+\cos(2\theta)}{2} \; ; \qquad\qquad \cos^2 \theta = \frac{1+\cos(2\theta)}{2}$$

The above expression becomes

$$I_u = \frac{1 + \cos 2\theta}{2} I_x + \frac{1 - \cos 2\theta}{2} I_y - \frac{1}{2} \sin 2\theta . I_{xy}$$

By following the same reasoning, we obtain:

$$I_v = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y) \cos 2\theta + I_{xy} \sin 2\theta \qquad\qquad 2.22$$

$$I_{uv} = \frac{1}{2}(I_x - I_y) \sin 2\theta + I_{xy} \cos 2\theta \qquad\qquad 2.23$$

We notice that

$$I_x + I_y = I_u + I_v \qquad\qquad 2.24$$

This means that the sum of the quadratic moments about two perpendicular axes remains constant whatever the value of the rotation angle θ.

We also notice that I_u, and I_v oscillate around the mean value $\frac{I_x+I_y}{2}$

By differentiating I_u, and I_v with respect to θ, we obtain:

$$-\frac{dI_u}{d(2\theta)} = +\frac{dI_v}{d(2\theta)}$$

The extrema are given for:

$$\frac{d}{d(2\theta)} = 0$$

Where

$$\tan 2\theta = -\frac{2I_{xy}}{I_x - I_y} \qquad\qquad 2.25$$

This relationship is satisfied for two values of θ between 0 and π, which correspond to a maximum I_1 (I_{max}) and a minimum I_2 (I_{min}); these are the principal moments of inertia. The axes corresponding to the principal moments of inertia are called the principal axes.

To determine I_1 (I_{max}) and I_2 (I_{min}), we can use Mohr's circle. To construct Mohr's circle, follow these steps:

1. Draw an orthogonal and orthonormal coordinate system (O, I_Q, I_{QR}) (Fig. 2.16).
2. Plot the points A (I_x, I_{xy}) and B (I_y, $-I_{xy}$) in this coordinate system.
3. Determine point C, the intersection of line AB with the horizontal axis.
4. From Mohr's circle, deduce I_1 (I_{max}) and I_2 (I_{min}).

We have:

$$I_{max} = I_1 = \overrightarrow{OC} + R$$

$$I_{min} = I_2 = \overrightarrow{OC} - R$$

Where

$$I_{max} = \frac{I_x + I_y}{2} + \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(I_{xy}\right)^2} \qquad 2.26$$

$$I_{min} = \frac{I_x + I_y}{2} - \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + \left(I_{xy}\right)^2} \qquad 2.27$$
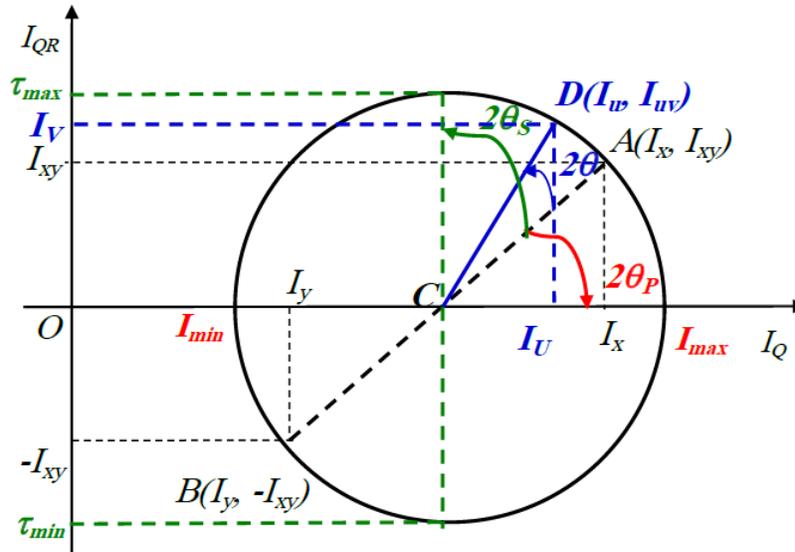
Fig 2.16. Mohr's Circle.

2.7. Modulus of Resistance

The moment of resistance of a cross-section is the ratio of the axial moment of inertia to the furthest distance from that axis.

$$W_x^{min} = \frac{I_x}{y^{max}} ; \qquad W_y^{min} = \frac{I_y}{x^{max}} \qquad\qquad 2.28$$

- **Example 2.8**

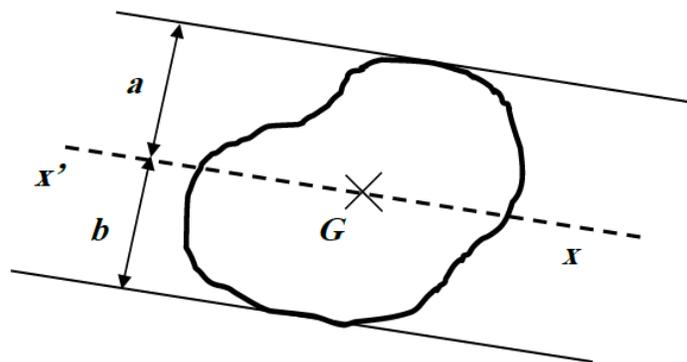For the following figure, determine the minimum moment of resistance.
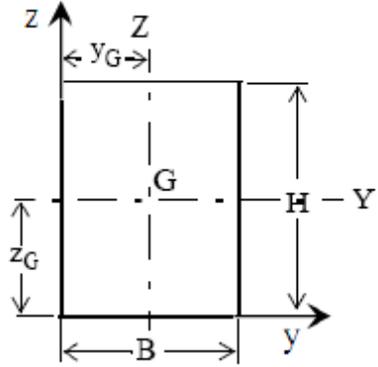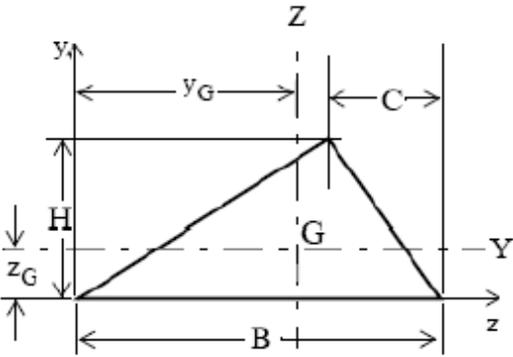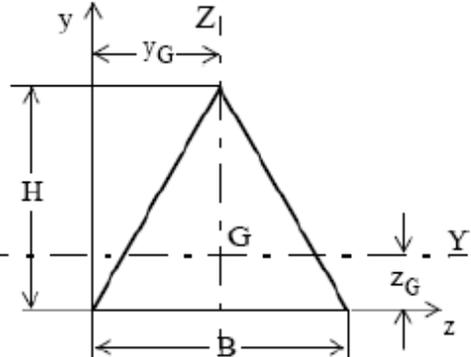

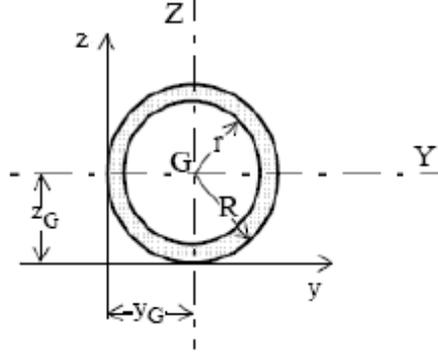**Fig 2.17**

- Si $a < b \Rightarrow W_x^{min} = I_x/b$
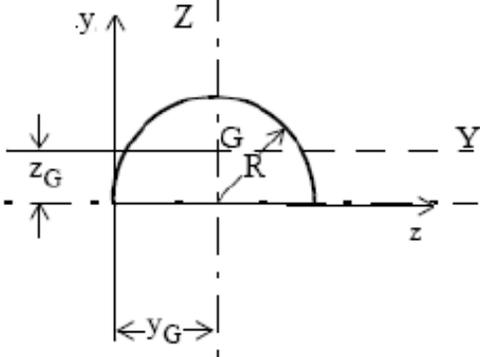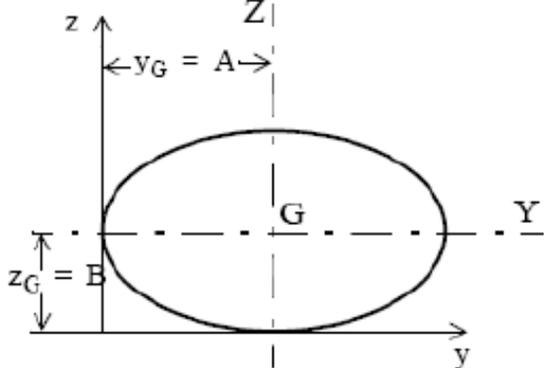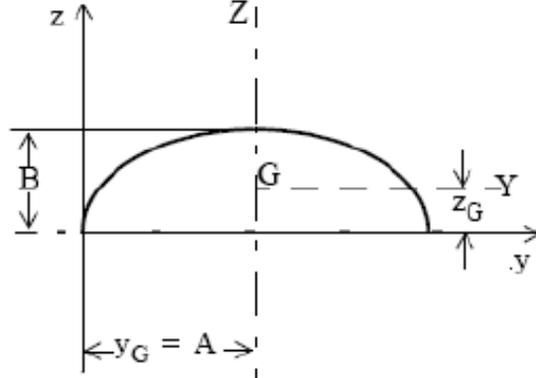- Si $a > b \Rightarrow W_x^{min} = I_x/a$

2.8. Radius of Gyration

The radius of gyration of an area A about the x-axis or y-axis is defined by:
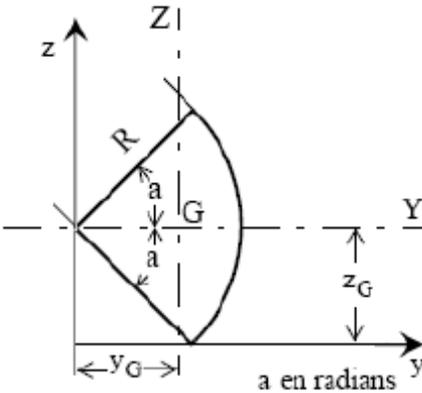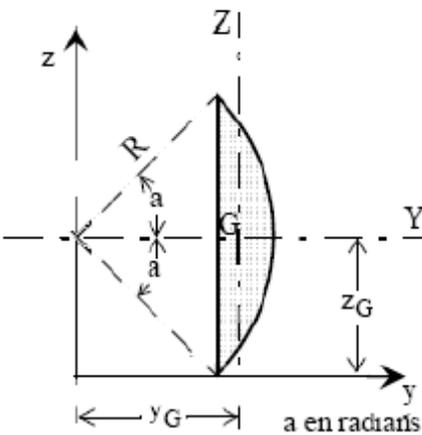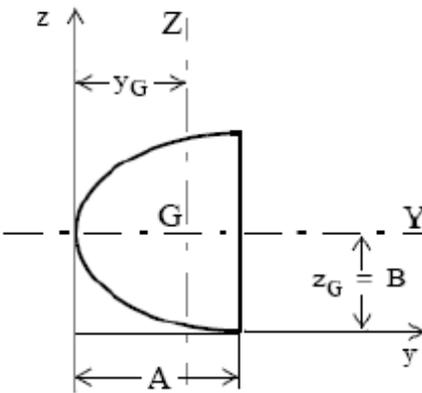
$$5. \quad i_x = \sqrt{\frac{I_x}{A}} \quad ; \quad i_y = \sqrt{\frac{I_y}{A}} \qquad\qquad 2.29$$

The geometric properties of the standard cross-section are presented in Appendix

Section	Aire / Coordonnées du centre de gravité	Moments quadratiques
Rectangle	$A = B.H$ $y_G = \dfrac{B}{2}$ $z_G = \dfrac{H}{2}$	$I_Y = \dfrac{BH^3}{12}$ $I_Z = \dfrac{HB^3}{12}$
Triangle quelconque	$A = B.H / 2$ $y_G = \dfrac{2B+C}{3}$ $z_G = \dfrac{H}{3}$	$I_Y = \dfrac{BH^3}{36}$ $I_Z = \dfrac{BH}{36} \cdot (B^2 + C^2 - BC)$
Triangle équilatéral	$A = B.H / 2$ $y_G = \dfrac{B}{2}$ $z_G = \dfrac{H}{3}$	$I_Y = \dfrac{BH^3}{36}$ $I_Z = \dfrac{B^3 H}{48}$

Trapèze isocèle	$A = (A + B).H / 2$	
	$y_G = \dfrac{B}{2}$ $^zG_a = \dfrac{H \cdot (B + 2A)}{3 \cdot (B + A)}$	$I_Y = \dfrac{H^3 \cdot (A^2 + 4AB + B^2)}{36 \cdot (A + B)}$ $I_Z = \dfrac{H \cdot (A + B) \cdot (A^2 + B^2)}{48}$
Cercle	$A = \pi R^2$ $y_G = R$ $^zG = R$	$I_Y = \dfrac{\pi \cdot R^4}{4}$ $= 0.0491 \cdot D^4$ $I_Z = \dfrac{\pi \cdot R^4}{4}$ $= 0.0491 \cdot D^4$
  $D = 2 \cdot R$		
Cercle creux (tube)	$A = \pi (R^2 - r^2)$ $y_G = R$ $^zG = R$	$I_Y = \dfrac{\pi \cdot (R^4 - r^4)}{4}$ $I_Z = \dfrac{\pi \cdot (R^4 - r^4)}{4}$
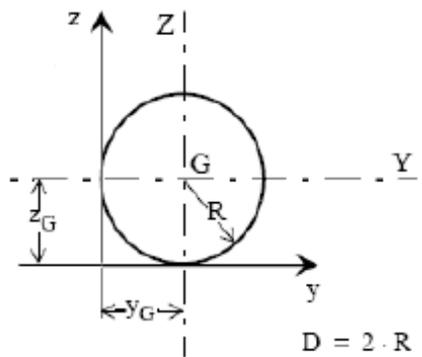
Demi-cercle 	$A = \pi R^2 / 2$ $y_G = R$ $z_G = 0.2122 \cdot D$ $= 0.4244 \cdot R$	$I_Y = 0.1098 \cdot R^4$ $I_Z = 0.3927 \cdot R^4$
Ellipse 	$\text{Aire} = \pi.A.B$ $y_G = A$ $z_G = B$	$I_Y = \dfrac{\pi \cdot AB^3}{4}$ $= 0.7854 \cdot AB^3$ $I_Z = \dfrac{\pi \cdot A^3 B}{4}$ $= 0.7854 \cdot A^3 B$
Demi-ellipse 	$\text{Aire} = \pi.A.B / 2$ $y_G = A$ $z_G = 0.424 \cdot B$	$I_Y = 0.1098 \cdot AB^3$ $I_Z = 0.3927 \cdot A^3 B$

Secteur circulaire	$A = (\pi/360°).\varphi.R^2$	$I_Y = \dfrac{R^4}{4} \cdot (a - sina\,cosa)$
	$y_G = \dfrac{2}{3} \cdot \dfrac{R sina}{a}$	
	$z_G = R sina$	$I_Z = \dfrac{R^4}{4} \cdot \left(a - \dfrac{16(sima)^2}{9a}\right.$
		$\left. + \dfrac{sin2a}{2}\right)$
$\varphi = 2.a$ (en degré)		
Segment circulaire	$A = \dfrac{R^2}{2}(2a - sin2a)$	$I_Y = \dfrac{AR^2}{4}\left(1 - \dfrac{2(sina)^3 cosa}{3(a - sina\,cosa)}\right)$
	$y_G = \dfrac{4R(sina)^3}{3(2a - sin2a)}$	$I_Z = \dfrac{AR^2}{4}\left(1 + \dfrac{2(sina)^3 cosa}{a - sina\,cosa}\right)$
	$z_G = R sina$	$- \dfrac{4R^6(sina)^6}{9A}$
Segment parabolique	$A = (2/3).(2B.A)$	$I_Y = 0.0396 \cdot AB^3$
	$y_G = 0.6 \cdot A$	$I_Z = 0.0457 \cdot A^3 B$
	$z_G = 0.375 \cdot B$	