

1.1. Objectives and Hypotheses of Strength of Materials

1.1.1. Definitions

- **Strength of Materials**

Strength of Materials (SM) is a branch of continuum mechanics that deals with the deformation of structures such as machines in mechanical engineering or buildings in civil engineering. It is an experimental science concerned with real solids. This discipline allows us to study the resistance of mechanical components, the mechanical actions applied to them, and the resulting deformations. To do so, it is first necessary to properly model the various possible mechanical connections and the external forces acting on the system.

Statics, on the other hand, is a branch of mechanics that studies the conditions under which a body remains in equilibrium, taking into account the forces exerted on it by its external environment.

- **Concept of Stress**

A stress is a force per unit area that acts within a material.

Consider a solid body Ω subjected to forces (concentrated or distributed), as illustrated in Figure 1.1-a.

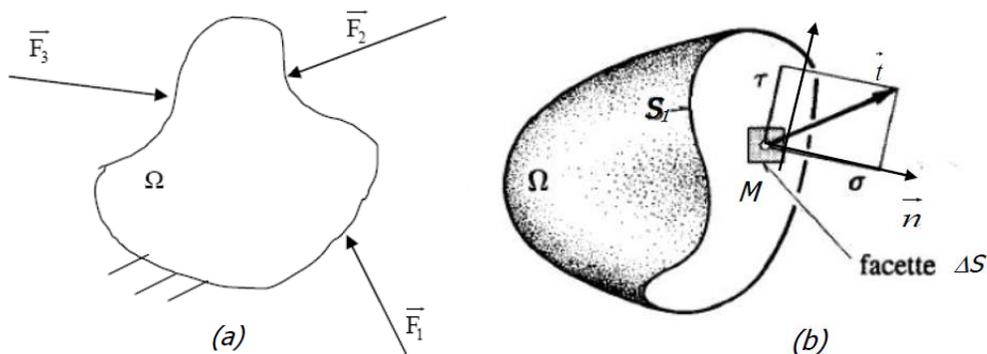


Fig 1.1. Schematization of a Loaded Solid

We cut the solid Ω into two parts, S_1 and S_2 .

Let us consider a point M surrounded by a surface ΔS .

The solid S_2 exerts a mechanical action on the solid S_1 , denoted $\overline{\nabla F}_{S_2/S_1}$, which can be modeled as a distributed force.

Thus, we have:

$$\overline{\nabla F}_{S_2/S_1} = \vec{C}(M, \vec{n}) \nabla S \quad 1.1$$

The vector $\vec{C}(M, \vec{n})$ is called the stress vector at point M with normal \vec{n} (where \vec{n} is the unit normal vector to the surface ΔS , directed outward).

The stress vector can be decomposed along the vectors \vec{n} and \vec{t} (where \vec{t} is a unit vector lying in the tangent plane to ΔS) as shown in Figures 1.1-b and 1.2 in the following form:

- σ is called the normal stress.
- τ is called the shear stress.

Both the normal stress and the shear stress are expressed in (Pa) or (MPa).

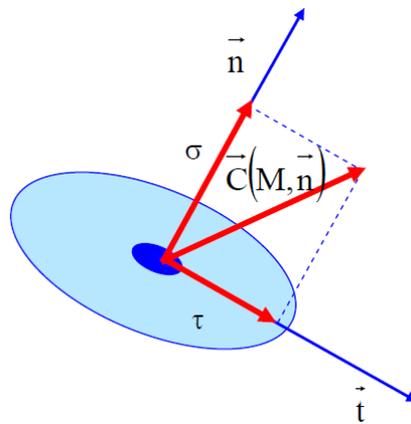


Fig 1.2. Decomposition of the Stress Vector along the Normal \vec{n} and the shear \vec{t}

Experimentally, for each material, an allowable stress limit, denoted $[\sigma]$, is defined. Beyond this value, the component undergoes deterioration in its mechanical properties, dimensions, or may even experience failure.

The purpose of strength of materials analysis is to ensure that the stresses produced by external loads do not exceed the material's allowable stress limit $[\sigma]$.

- A stress is a calculation tool; it cannot be directly observed, but its effects can be observed for example, through the study of deformations.
- Since stress is the ratio of force to surface area, the factors that directly influence it are the applied loads and the cross-sectional area of the component.
- **Concept of Deformation**

Every solid subjected to a force undergoes deformation. Deformations result from and vary with the applied loads on the object. They are evidenced by changes in dimensions and can be either elastic or plastic.

- Elastic deformation occurs when the solid returns to its original shape after the forces are removed (as in the case of a normally loaded spring).
- Plastic deformation occurs when the solid remains deformed even after the forces are removed (as in the case of modeling clay).

It should be noted that no material is perfectly elastic. However, deformation is generally elastic for sufficiently small loads and becomes plastic once the stress exceeds a certain threshold σ_e , called the elastic limit.

- The elastic limit is a characteristic stress of the material; it does not depend on the dimensions of the component or on the applied loads.

In the study of strength of materials, we will focus exclusively on elastic materials. This means that we will always assume the applied loads on the studied structures are small enough for the resulting deformations to remain elastic.

1.2. Hypotheses

The Strength of Materials (SM) requires simplifying hypotheses regarding the geometry and materials of the objects studied for its correct application.

1.2.1. Geometry

The most studied solid in SM is the beam, which is the subject of this course. A beam is a solid generated by a plane surface (S) (fig. 1.3) whose center of gravity describes a curve called the neutral axis, which must satisfy the following two conditions:

- Its radius of curvature must be at least 5 times greater than the largest dimension of (S).
- Its length must be at least 5 times greater than the largest dimension of (S).

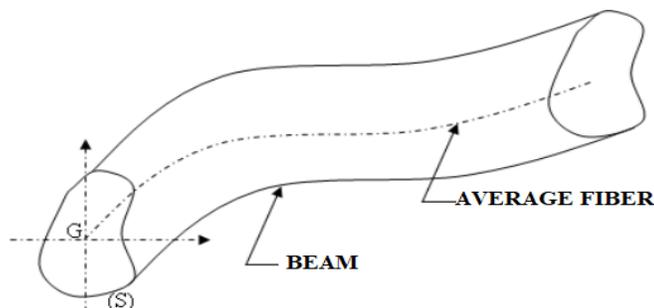


Fig 1.3. A beam

In the general case, (S) can vary along the average fiber. If (S) is constant, then the beam is said to have a constant section. (Fig. 1.4)

In the general case, the average fiber is curved. If the average fiber is a straight line, the beam is said to be straight.

The "average fiber" of a beam refers to an imaginary line that runs along the length of the beam, representing the neutral axis where the material experiences no tension or compression during bending. This concept is essential in understanding how beams deform under loads. The average fiber helps in analyzing stress distribution and determining how the beam will behave structurally.

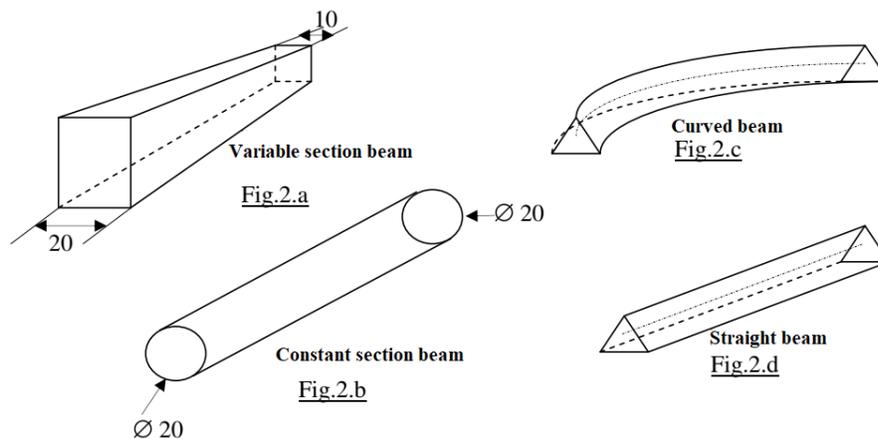


Fig 1.4. Special Cases

1.2.2. Material

The solids studied in Strength of Materials (SM) must be made from homogeneous, isotropic, and linear elastic materials.

- **Continuity**

The material is assumed to be continuous, meaning that the distances between molecules are always very small. At the scale of strength of materials (SM), the material therefore appears continuous.

- **Homogeneity**

A material is considered homogeneous "at a certain scale" if it exhibits constant properties at each point at that scale.

- **Isotropy**

A material is said to be isotropic if it possesses the same properties in all directions.

1.3. Additional Assumptions

In addition to the stated assumptions, the solids studied must meet the following additional criteria:

- **Navier-Bernoulli Assumptions:** The plane sections along the average fiber must remain flat after deformation.

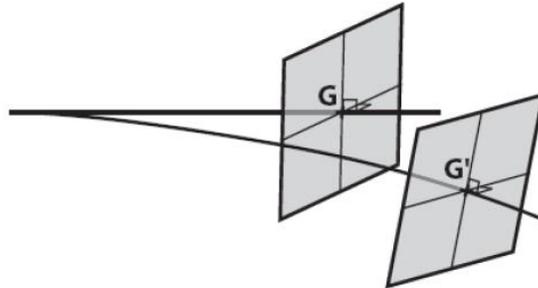


Fig 1.5. Schematization of the Navier-Bernoulli hypothesis

- **Saint-Venant Principle:** The results obtained from strength calculations only apply at a sufficiently distant location from the concentrated force application areas and joints.

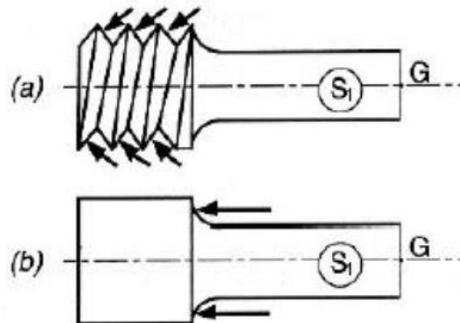


Fig 1.6. Schematization of the Barré de Saint-Venant hypothesis

- **Small Deformation Assumption:** The deformations must be small compared to the dimensions of the beam.

1.4. Different Types of Loading

Loads can be classified in various ways. Two main types of mechanical actions (loads) are generally distinguished:

- Mechanical contact actions (such as contact connections between solids, pressure, etc.);
- Mechanical actions at a distance (such as gravitational fields, electromagnetic forces, etc.).

The first type of action is applied on the surface of the solid (surface action), while the second acts within its volume (volumetric action).

We also distinguish between external and internal actions in a system of solids:

- External forces (or actions) applied to an isolated material system are all the mechanical actions acting on this system whose origin lies outside the system. These actions can be either mechanical contact actions or distance actions (such as gravity).
- Internal forces are the forces that the different parts of the isolated system exert on each other.

❖ Modeling of Mechanical Actions

The analysis of mechanical actions can only be performed using models that depict the actions and their effects on the solid. Two primary models are used to represent and study mechanical actions: the local and the global models.

The local model (Fig. 1.7) allows the study of the action and its effects at every point in the region where it is applied for example, studying contact pressures, stresses in materials, or deformation of the solid.

In the global model (Fig. 1.8), a mechanical action is represented by a Torsor (called the Mechanical Action Torsor). This model eliminates the local effects of the action but makes its use convenient for studying equilibrium or dynamics.

These two models global and local, are not interchangeable. While the mechanical action torsor can be determined from the regional distribution of forces, the reverse process cannot be done without making assumptions about that distribution.

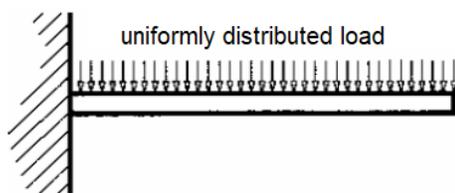


Fig 1.7. Local model

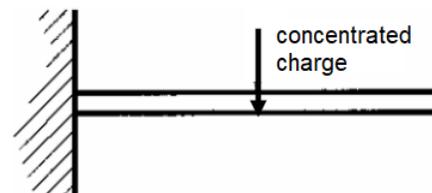


Fig 1.8. Global model

The uniformly distributed load (Fig. 1.7) is replaced by the equivalent resultant force \vec{F} (Fig. 1.8).

1.5. Definition of the Torsor

The complete definition of an effort (force) involves two vectors:

- A force \vec{R} , called the resultant;
- A moment $M(\vec{R})/o$, taken about an arbitrary point O, called the moment.

These two vectors, known as the reduction elements, can be combined into a single expression using a new mathematical tool called a torsor.

A torsor is denoted by $\{\vec{\tau}\}$, and its reduction elements at point O are written as $\{\vec{\tau}\}_O$.

1.6. Types of Supports in Civil Engineering

External actions (external forces) applied to solids can be known or unknown. Among the known forces, we include those representing the self-weight of elements, climatic actions (wind, snow, waves). These actions are specified in the design specifications of the building such as the weight of machinery, the action of overhead cranes, and the use of rooms or spaces.

The unknown forces arise from the connections (supports) between the studied solid and the load-transmitting elements. These connections are used to restrict certain degrees of freedom (DOF) of the solids. Our analysis will be carried out in the context of planar mechanics and civil engineering. Connections, in restricting displacements, generate unknown forces called reaction forces (or simply reactions). Each connection is associated with a force torsor, which depends on its kinematic characteristics.

The elementary possible motions in a plane are:

- Two translations (Δ_x and Δ_y),
- One rotation ($\Omega = \Omega_k$).

The main types of supports in civil engineering are:

- Simple support: 1 DOF blocked \rightarrow (1 unknown reaction)
- Elastic support: 1 DOF controlled \rightarrow (1 unknown reaction and a behavior law)
- Double Support (Hinge): 2 DOF blocked \rightarrow (2 unknown reactions)
- Fixed Support: 3 DOF blocked \rightarrow (3 unknown reactions)

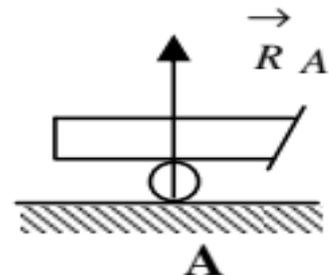
1.6.1. Simple Support

A simple support is characterized by 2 degrees of freedom and 1 component of reaction.

The two degrees of freedom are:

- Rotation around the support (Ω),
- Translation parallel to the support (Δ_x).

The reaction is known by its point of application (the contact point with the support) and its direction (perpendicular to the support). Only the intensity remains to be determined.

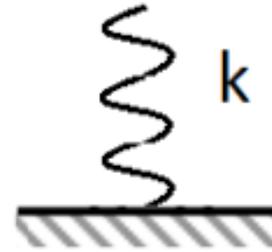


1.6.2. Elastic Support

The elastic support controls a translation through the known stiffness (k) of the support device. It follows a behavioral relationship of the form:

$$F_y = k\Delta_y$$

This type of support allows a controlled translation Δ_y , may or may not allow a horizontal translation Δ_x (in the case of a sliding support), and allows rotation (Ω).



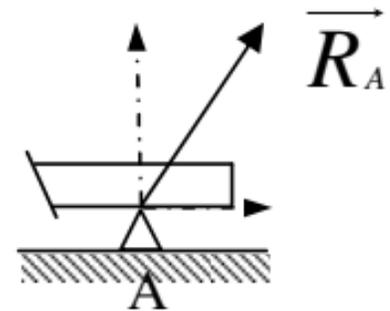
1.6.3. Double Support (Hinge)

A double support or hinge is characterized by 1 degree of freedom and two components of reaction.

The degree of freedom is:

- Rotation around the support.

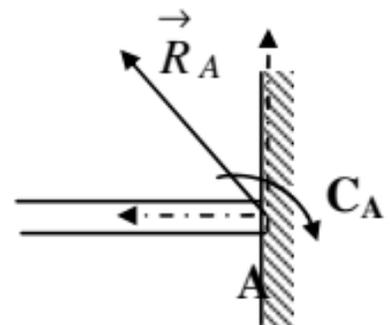
The reaction of the support is known only by its point of application (the contact point with the support). The reaction is decomposed into two perpendicular directions, with both components needing to be determined.



1.6.4. Fixed Support

A fixed support is characterized by no degrees of freedom and three components of reaction:

- Two components along two perpendicular directions passing through point A,
- A couple applied at point A.



1.7. General Principle of Equilibrium – Equilibrium Equations

1.7.1. Statement of the Principle

Let a solid body (S) be subjected to a system of external forces represented by the torsor $\{\vec{F}_{ext}\}$.

Let $\{\vec{\mathcal{R}}\}$ be the reference frame associated with (S); the solid (S) is in equilibrium if and only if:

$$\{\vec{F}_{ext}\} = (\vec{0}) \quad 1.2$$

1.7.2. Practical Applications

The equality of two torsors implies the equality of their reduction components.

Let O be the chosen point:

$$\{\vec{F}_{ext}\}_o = (\vec{0})_o \Rightarrow \begin{cases} \vec{R}_{(\vec{F}_{ext})} = \vec{0} \\ \vec{M}_{(\vec{F}_{ext})/o} = \vec{0} \end{cases} \quad \begin{matrix} 1.3 \\ 1.4 \end{matrix}$$

Equations (3) and (4) are two vector equations that provide:

- 6 scalar equations in space.
- 3 scalar equations in a plane.

1.8. Principle of Section (or Isolation) – Reduction Elements

Let there be two solids, (S₁) and (S₂), and (S) the system formed by (S₁) and (S₂) as shown in Figure 3.2.

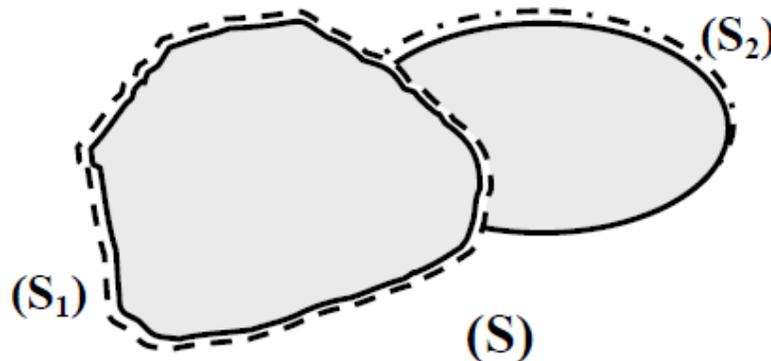


Fig 1.9. Illustration of the boundaries of a solid.

Let the torsor of external actions on (S) be:

$$\{\vec{F}\} = \{\vec{F}_1\} + \{\vec{F}_2\} \quad 1.5$$

$\{\vec{F}_1\}$ is the torsor of the actions applied to the external boundary of (S₁)

$\{\vec{F}_2\}$ is the torsor of the actions applied to the external boundary of (S₂)

Let us summarize the forces acting on (S₁). When isolating (S₁), we have:

$$\{\vec{D}_1\} = \{\vec{F}_1\} + \{\vec{F}_{2/1}\} \quad 1.6$$

Where

$\{\vec{F}_{2/1}\}$ is the torsor of the actions exerted by (S₂) on (S₁) across the common boundary

- **Principle of Reciprocal Actions**

If we now isolate (S₂), the summary of external actions gives:

$$\{\vec{D}_2\} = \{\vec{F}_2\} + \{\vec{F}_{1/2}\} \quad 1.7$$

Where

$\{\vec{F}_2\}$ is the torsor of the actions applied to the free boundary of (S₂)

$\{\vec{F}_{1/2}\}$ is the torsor of the actions exerted by (S₁) on (S₂)

We have:

$$(S) = (S_1) \cup (S_2) \quad 1.8$$

Therefore:

$$\{\vec{F}\} = \{\vec{D}_1\} + \{\vec{D}_2\} = (\{\vec{F}_1\} + \{\vec{F}_{2/1}\}) + (\{\vec{F}_2\} + \{\vec{F}_{1/2}\}) = (\{\vec{F}_1\} + \{\vec{F}_2\}) \quad 1.9$$

Let:

$$\{\vec{F}_{2/1}\} + \{\vec{F}_{1/2}\} = \vec{0} \quad 1.10$$

Equation (1.10) represents the principle of reciprocal actions. In simplified form, the principle of reciprocal or mutual actions for two solids in contact is written as:

$$\vec{F}_{2/1} = -\vec{F}_{1/2} \quad 1.11$$

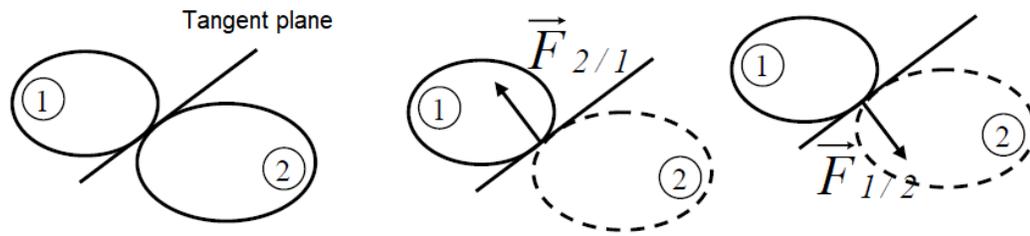


Fig 1.10. Illustration of the principle of reciprocal actions

1.9. Definitions and sign conventions of internal forces

The internal forces at a point G on the mean line of a beam are the components of the reduction elements of the internal forces torsor. These internal forces are denoted as follows (Fig. 1.10):

- N is the normal force (in the x direction)
- T_y is the shear force in the y direction
- T_z is the shear force in the z direction
- $T = T_y + T_z$ is the resultant shear force
- M_t is the torsional moment (about the x -axis)
- M_y is the bending moment (about the y -axis)
- M_z is the bending moment (about the z -axis)
- $M = M_y + M_z$ is the resultant bending moment

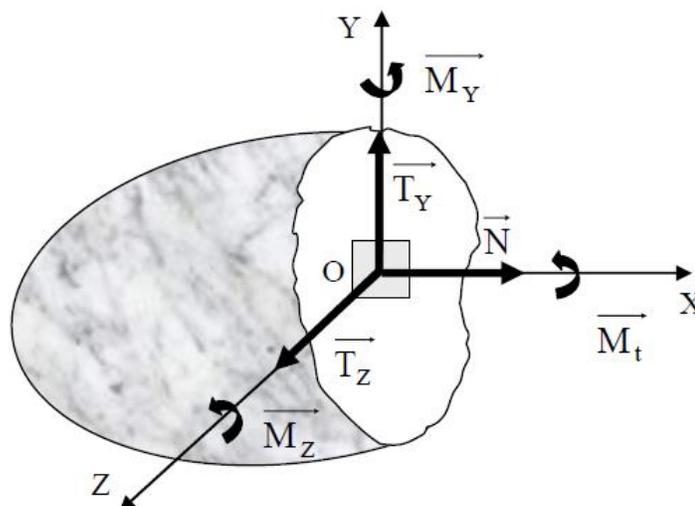


Fig 1.10. Internal forces at a point along the mean line of a beam

- **Diagram of internal forces**

The internal force diagrams are the curves representing the variation of each internal force along the mean line of the beam. These diagrams are useful for quickly identifying the most stressed sections.

- **Simple loadings**

The most commonly encountered types of loading are tension or compression, bending, torsion, and shear. Some examples of simple loadings are given in Table 1.1, and Figure 1.11 illustrates these types of loadings.

Table 1.1. Some types of loadings

Nature of the loads	Values of the components			
	Normal force	Shear force	Torsional moment	Bending moment
Tension ($N > 0$) Compression ($N < 0$)	$N \neq 0$	$T_y = 0$ $T_z = 0$	$M_t = 0$	$M_{fy} = 0$ $M_{fz} = 0$
Simple shear	$N = 0$	$T_y \neq 0$ and/where $T_z \neq 0$	$M_t = 0$	$M_{fy} = 0$ $M_{fz} = 0$
Simple torsion	$N = 0$	$T_y = 0$ $T_z = 0$	$M_t \neq 0$	$M_{fy} = 0$ $M_{fz} = 0$
Simple bending	$N = 0$	$T_y = 0$ $T_z = 0$	$M_t = 0$	$M_{fy} \neq 0$ and/where $M_{fz} \neq 0$

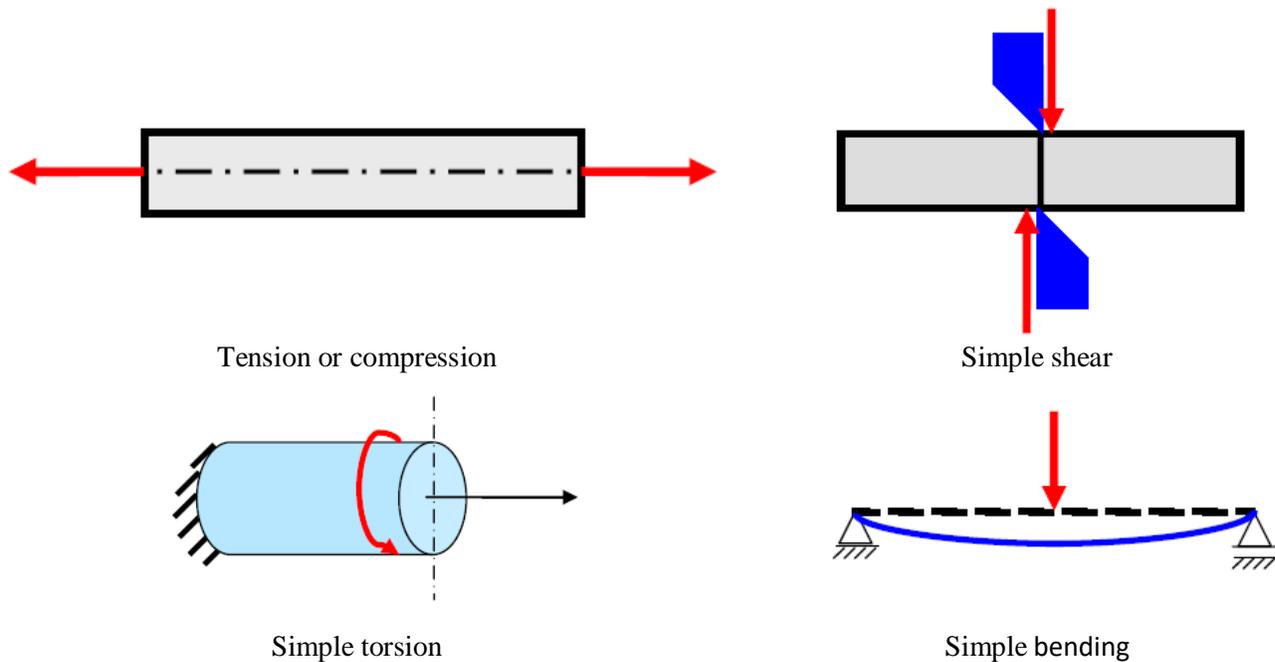


Fig 1.11. Beam subjected to a simple loading