University of Mila

Institute of Nature and life Sciences

Common Core Departement of NLS

Mathematics Statistics Informatics

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Chapter 1

Real functions of real variable Part II

Contents

1.1 Integrals and Primitives

1.1.1 Primitives

Definition 1.1.1: Let $f : [a, b] \longmapsto \mathbb{R}$ be a function, we say that the function F is primitive function of f over $[a, b]$ if and only if F is differentiable on $[a, b]$ and

$$F'(x) = f(x), \forall x \in [a, b].$$

Example 1.1.1 *The function* $F(x) = \frac{1}{3}x^3$ *is a primitive of the function* $f(x) = x^2$ *over* $\mathbb{R}$ *because*

$$\forall x \in \mathbb{R} : F'(x) = \left(\frac{1}{3}x^3\right)' = x^2 = f(x).$$

Definition 1.1.2: The set of all primitives of the function $f : [a, b] \longmapsto \mathbb{R}$ is called the indefinite integral of f, denoted $\int f(x)dx$, so if F is a primitive of f on $[a, b]$, we have

$$\int f(x)dx = F(x) + c, c \in \mathbb{R}.$$

Example 1.1.2 *For all* $x \in [1,2] : \int \frac{1}{x}dx = \ln(x) + c, c \in \mathbb{R}.$

Theorem 1.1.1 *Every continuous function on* $[a,b]$ *admits a primitive on* $[a,b]$.

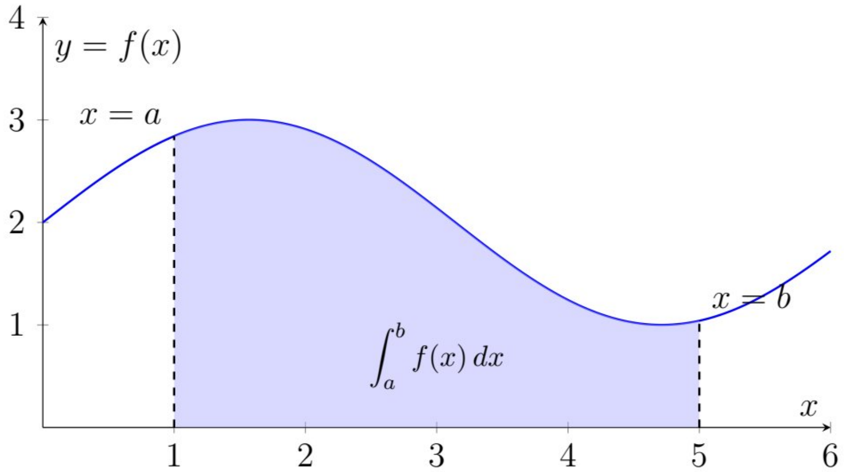
1.1.2 Integrals.

1.1.2.1 Definition from area.

a) **Case of positive function.**

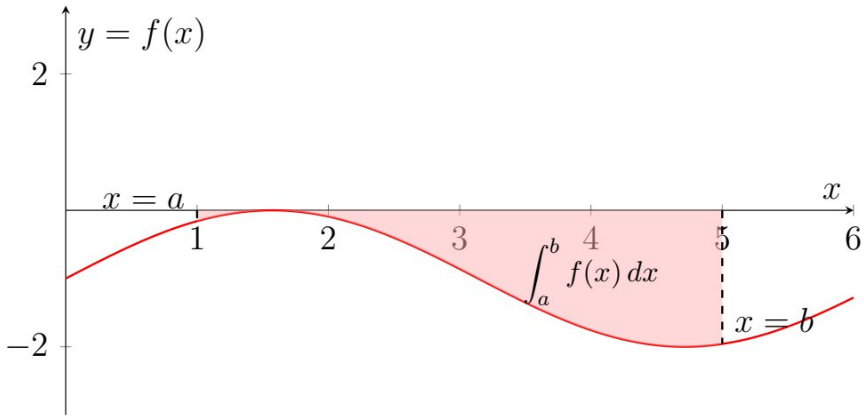
Definition 1.1.3: Let f be a positive and continuous function on an interval $[a,b]$, we call the integral of f between a and b, the area A of the region the curve of the function f and the lines $x = a$ and $x = b$. We denote

$$A = \int_a^b f(x)dx.$$



b) **Case of any function.**

If f is negative, then $-f$ is positive so $\int_a^b f(x)dx$ is being the opposite of the area defined above as $A = \int_a^b -f(x)dx$.



If we want to find the area for a function f without knowing its sign, we can use the formula

$$\int_a^b |f(x)|\,dx.$$

1.1.2.2 Definition from primitive.

> **Definition 1.1.4:** Let f be a continuous function on $[a, b]$ and F one of its primitives, we call the integral of f between a and b the quantity
>
> $$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

Example 1.1.3 *Calculation of the integral* $\int_2^3 xdx$*:*
The primitive of $f(x) = x$ *is* $F(x) = \frac{x^2}{2}$ *so* $\int_2^3 xdx = F(3) - F(2) = \frac{9}{2} - \frac{4}{2} = \frac{5}{2}$.

1.1.2.3 Properties of the integral:

Let $f : [a, b] \longmapsto \mathbb{R}$ be a continuous function and $c \in \,]a, b[$ then

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
2. $\int_a^a f(x)dx = 0$.
3. Charles relation.
$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

1.1.2.4 Integral and operations.

Linearity:

Let f, g be two continuous functions on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$ then

$$\int_a^b \alpha f(x) + \beta g(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx.$$

Positivity:

Let $f : [a, b] \longmapsto \mathbb{R}$ be a continuous function then

$$f \text{ is positive} \Longrightarrow \int_a^b f(x)dx \geqslant 0.$$

Increasing of the integral:

Let f, g be two continuous functions on $[a, b]$ then

$$f(x) \leq g(x) \Longrightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Remark 1.1.1

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)| \, dx.$$

1.1.2.5 Integration methods.

1.1.2.6 Direct integration.

This method is based on the properties of integrals and the using of usual primitives tables

Fonction $f(x)$	**Primitive** $F(x)$	**I**
k	$kx + C$	$\mathbb{R}$
$x^n,\ n \in \mathbb{N}^*$	$\dfrac{x^{n+1}}{n+1} + C$	$\mathbb{R}$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$	$]-\infty, 0[\cup]0, +\infty[$
e^x	$e^x + C$	$\mathbb{R}$
$a^x,\ a > 0,\ a \neq 1$	$\dfrac{a^x}{\ln a} + C$	$\mathbb{R}$
$\sin x$	$-\cos x + C$	$\mathbb{R}$
$\cos x$	$\sin x + C$	$\mathbb{R}$
$\frac{1}{\cos^2 x} = 1 + \tan^2 x$	$\tan x + C$	$]-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi[, k \in \mathbb{Z}$

Example 1.1.4 *Compute*

$$\int\left(\cos^2(x) + \frac{1}{1+x^2} + \sqrt{x}\right) dx$$

We have

$$\cos(2x) = 2\cos^2(x) - 1 \Longrightarrow \cos^2(x) = \frac{1}{2}\cos(2x) - \frac{1}{2}.$$

We find that

$$\begin{aligned}
\int\left(\cos^2(x) + \frac{1}{1+x^2} + \sqrt{x}\right) dx &= \int\left(\frac{1}{2}\cos(2x) - \frac{1}{2}\right) dx + \int \frac{1}{1+x^2}dx + \int \sqrt{x}dx \\
&= \frac{1}{4}\sin(2x) + \frac{1}{2}x + c_1 + \arctan(x) + c_2 + \frac{2}{3}x^{\frac{3}{2}} + c_3 \\
&= \frac{1}{4}\sin(2x) + \frac{1}{2}x + \arctan(x) + \frac{2}{3}x^{\frac{3}{2}} + c,\ \ c \in \mathbb{R}.
\end{aligned}$$

1.1.2.7 Integration by substitution (Change of variables).

Proposition 1.1.1 *Let f be a continous function on $[a,b]$, and let g continuously differentiable function on interval $[c,d]$ such that $g([c,d]) = [a,b]$, then we have*

$$\int_a^b f(x)dx = \int_c^d f(g(t))g'(t)dt$$

Example 1.1.5 *Calculate*

❶$I = \int \frac{x}{\sqrt{1+x^2}}dx.$

We put $t = 1 + x^2 \Longrightarrow dt = 2xdx.$

Replacing in the integral I gives us:

$$I = \int \frac{1}{2\sqrt{t}}dt = \sqrt{t} + c \ /c \in \mathbb{R}.$$

$$\Longrightarrow I = \sqrt{1+x^2} + c \ /c \in \mathbb{R}.$$

❷ $J = \int \frac{1}{\cosh(x)}dx.$

We have

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \Longleftrightarrow \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x}+1}.$$

Then

$$J = \frac{2e^x}{e^{2x}+1}dx,$$

by the change of variable $t = e^x \Longrightarrow dt = e^x dx$.

Then

$$\begin{aligned} J &= 2\int \frac{1}{1+t^2}dt \\ &= 2\arctan(t) + c, \ c \in \mathbb{R} \\ &= 2\arctan(e^x) + c, \ c \in \mathbb{R} \end{aligned}$$

1.1.2.8 Integration by parts.

Proposition 1.1.2 *Let f and g be two differentiable functions on $[a,b]$, then we have*

$$\int_a^b f'(x)g(x)dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)dx.$$

Example 1.1.6 *Calculate the following integral by parts*

❶ $\int xe^{-2x}dx$.

We pose

$$\begin{cases} f(x) = x & \longrightarrow f'(x) = 1 \\ g'(x) = e^{-2x} & \longrightarrow g(x) = -\frac{1}{2}e^{-2x} \end{cases}$$

According to the integration by parts formula, we obtain

$$\begin{aligned} \int_0^1 xe^{-2x}dx &= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx, \\ &= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c,\ c \in \mathbb{R}. \end{aligned}$$

❷ $\int \sqrt{1-x^2}dx$ *on* $]-1,1[$.

Let's put

$$\begin{cases} f(x) = \sqrt{1-x^2} & \longrightarrow f'(x) = \frac{-x}{\sqrt{1-x^2}} \\ g'(x) = 1 & \longrightarrow g(x) = x. \end{cases}$$

Then

$$\int \sqrt{1-x^2}dx = x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}dx$$

We have also

$$\int \frac{x^2}{\sqrt{1-x^2}}dx = \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + c, \quad c \in \mathbb{R}$$

So

$$\int \sqrt{1-x^2}dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + c, \quad c \in \mathbb{R}$$

Proposition 1.1.3 *Let f be a continuous function on $[a,b]$, then*

❶ *If f is a even function , then*

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

❷ *If f is a odd function , then*

$$\int_{-a}^{a} f(x)dx = 0$$