University of Mila

Institute of Nature and life Sciences

Common Core Departement of NLS

Mathematics Statistics Informatics

1^{st} **Year NLS, Semester 1**

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Chapter 1

Real functions of real variable Part I

Introduction

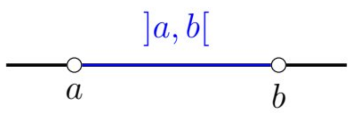
This chapter introduces the fundamental concepts of real functions of a real variable: definitions, domain, monotonicity, boundedness, limits, continuity, integrals and primitives.

1.1 Recalls and definitions:

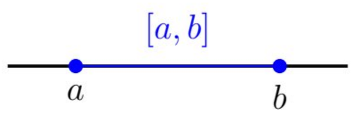
1.1.1 Intervals of $\mathbb{R}$

Let $a, b \in \mathbb{R}$ such that $a < b$, we call

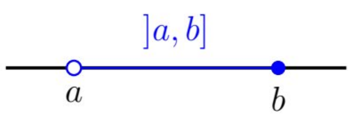
- **The open interval :** $]a, b[= \{x \in \mathbb{R} \mid a < x < b\}$



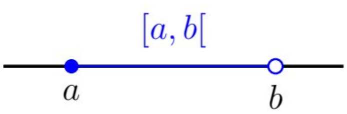
- **The closed interval :** $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$



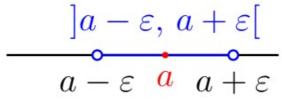
- **The semi-open interval :** $]a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$



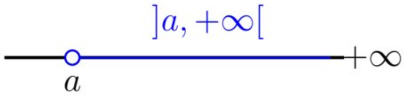
  $[a, b[= \{x \in \mathbb{R} \mid a \leq x < b\}$



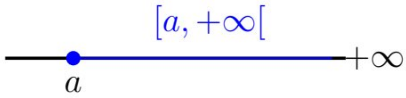
- **The open interval of center a the set :** $]a-\varepsilon, a+\varepsilon[$ with $\varepsilon > 0$



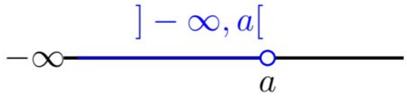
$]a, +\infty[= \{x \in \mathbb{R} \mid x > a\}$



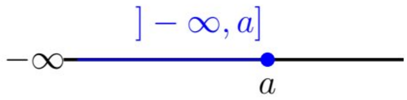
$[a, +\infty[= \{x \in \mathbb{R} \mid x \geq a\}$



$]-\infty, a[= \{x \in \mathbb{R} \mid x < a\}$



$]-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$



Notations

- $\mathbb{R}^* =]-\infty, 0[\cup]0, +\infty[$
- $\mathbb{R}_+ = \{x \in \mathbb{R} / x \geqslant 0\} =]0, +\infty[$
- $\mathbb{R}_- = \{x \in \mathbb{R} / x \leqslant 0\} =]-\infty, 0[$

Definition 1.1.1: A real function of a real variable is a map f of a part D of $\mathbb{R}$, we note

$$f : D \to \mathbb{R},$$

$$x \longmapsto f(x)$$

1.1.2 Domain of definition

Definition 1.1.2: The domain of definition of a function f denoted D_f is defined by

$$D_f = \{x \in \mathbb{R}, f(x) \text{ is defined}\}.$$

1.1.2.1 Practical determination of the domain of definition

Let f, g be two functions

1^{st} **cas**: function of type $\frac{f}{g}$ is defined for all $g \neq 0$

2^{nd} **cas**: function of type $\sqrt{x}$ is defined for all $f \geqslant 0$

3^{rd} **cas**: function of type $\frac{f}{\sqrt{g}}$ is defined for all $g > 0$

Example 1.1.1 ❶ $f(x) = \frac{x^2+2x+5}{x^2-1}$.

$$\begin{aligned} D_f &= \{x \in \mathbb{R},\ x^2 - 1 \neq 0\} \\ &= \{x \in \mathbb{R},\ (x-1)(x+1) \neq 0\} \\ &= \mathbb{R} - \{-1, 1\}. \end{aligned}$$

❷ $g(x) = \frac{3x+5}{\sqrt{2x-1}}$.

$$\begin{aligned} D_g &= \{x \in \mathbb{R},\ 2x - 1 > 0\} \\ &= \left\{x \in \mathbb{R},\ x > \frac{1}{2}\right\} \\ &= \left]\frac{1}{2}, +\infty.\right[\end{aligned}$$

1.1.3 General information on functions:

Let $f : D \to \mathbb{R}$ be a function and we suppose that $\forall x \in D, -x \in D$ we say that :

❶ f is **even**: if $f(-x) = f(x)$ for all $x \in D$.

The curve c_f of f is symmetric with respect to the axis (oy).

❷ f is **odd**: if $f(-x) = -f(x)$ for all $x \in D$.

The curve c_f of f is symmetric with respect to the origin $(0, 0)$.

❸ f is **periodic**: if there exists $T > 0$ such that $\forall x \in D, x+T \in D$ and $f(x+T) = f(x)$.

The smallest value of T is called the period of f.

❹ f is a **major** function $\Longleftrightarrow \exists M \in \mathbb{R},\ \forall x \in D,\ f(x) \leqslant M$

❺ f is a**minor** function $\Longleftrightarrow \exists m \in \mathbb{R},\ \forall x \in D,\ f(x) \geqslant m.$

❻f is a **bounded** function $\Longleftrightarrow \exists M, m \in \mathbb{R},\ \forall x \in D,\ m \leqslant f(x) \leqslant M.$

❼f is **increasing** function (resp strictly increasing) if $\forall x, y \in D \quad , x < y \Rightarrow f(x) \leqslant f(y),$ (resp $f(x) < f(y)$).

❽ f is **decreasing** function (resp strictly decreasing) if $\forall x, y \in D \quad , x < y \Rightarrow f(x) \geqslant f(y),$ (resp $f(x) > f(y)$).

❾ f is **monotonous** function if it is increasing or decreasing on D.

1.2 Limit of a function

1.2.1 Neighborhood of a Point x_0

Definition 1.2.1: We call neighborhood of a point x_0 any open interval of the form $]x_0 - \epsilon, x_0 + \epsilon[, \quad \epsilon > 0$

$$V_\epsilon(x) = \{x \in \mathbb{R}, x_0 - \epsilon < x < x_0 + \epsilon\}.$$

1.2.2 Function defined in a vicinity of a point

Definition 1.2.2: We vsay that a function $f : D \to \mathbb{R}$ is defined in the neighborhood of a point $x_0 \in \mathbb{R}$ if $\exists\ \epsilon > 0,\ ]x_0 - \epsilon, x_0 + \epsilon[\subseteq D$

1.2.3 Limits of a function at a point

Definition 1.2.3: We say that a function f defined in the vicinity of a point x_0 admits a limit $l \in \mathbb{R}$ as x tends $x_0$if

$$\forall \varepsilon > 0,\ \exists \sigma > 0,\ \forall x \neq x_0,\ |x - x_0| < \sigma \Rightarrow |f(x) - \ell| < \varepsilon.$$

We write $\lim\limits_{x \to x_0} f(x) = \ell$.

Proposition 1.2.1 *If f admits a limit at the point x_0 then this limit is unique and we have*

$$\lim_{x \to x_0} f(x) = \lim_{x \overset{>}{\longrightarrow} x_0} f(x) = \lim_{x \overset{<}{\longrightarrow} x_0} f(x) = \ell$$

1.2.4 Right-hand limit and left-hand limit

Definition 1.2.4:

1. We say that f has a right–hand limit ℓ at the point x_0, if

$$\forall \varepsilon > 0,\ \exists \sigma > 0,\ \forall x\ \ x_0 < x < x_0 + \sigma \Rightarrow |f(x) - \ell| < \varepsilon.$$

   and we write $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x) = \ell$.

2. We say that f has a left–hand limit ℓ at the point x_0, if

$$\forall \varepsilon > 0,\ \exists \sigma > 0,\ \forall x\ \ x_0 - \sigma < x < x_0 \Rightarrow |f(x) - \ell| < \varepsilon.$$

   and we write $\lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x) = \ell$.

Remark 1.2.1 *If* $\lim\limits_{x \overset{>}{\longrightarrow} x_0} f(x) \neq \lim\limits_{x \overset{<}{\longrightarrow} x_0} f(x)$ *then f does not admit alimit at point* x_0.

1.2.5 Operations on limits

Operations on Limits

Let f and $g : D \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be two functions, satisfying $\lim\limits_{x \to x_0} f(x) = \ell_1$ and $\lim\limits_{x \to x_0} g(x) = \ell_2$.

1. $\lim\limits_{x \to x_0} [f(x) + g(x)] = \ell_1 + \ell_2$.
2. $\lim\limits_{x \to x_0} [f(x) - g(x)] = \ell_1 - \ell_2$.
3. $\lim\limits_{x \to x_0} [f(x) \cdot g(x)] = \ell_1 \cdot \ell_2$.
4. $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\ell_1}{\ell_2}, \qquad$ if $\ell_2 \neq 0$.
5. $\lim\limits_{x \to x_0} [k \cdot f(x)] = \ell_1 k, \qquad k \in \mathbb{R}$.
6. $\lim\limits_{x \to x_0} |f(x)| = |\ell_1|$.
7. If $f(x) < g(x) \implies \ell_1 < \ell_2$.

1.2.6 Indeterminate forms

There are four algebraic indeterminate forms (IF)

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad \infty-\infty.$$

and three exponential forms

$$0^0, \quad \infty^0, \quad 1^\infty$$

.

Example 1.2.1 *Calculate the following limits*

❶ $\lim\limits_{x\to+\infty} -2x^3+4x-1 = +\infty-\infty \ (IF)$

$\lim\limits_{x\to+\infty} -2x^3+4x-1 = -2x^3\left(1-\frac{2}{x^2}+\frac{1}{2x^3}\right) = -\infty$

❷ $\lim\limits_{x\to1} \frac{x^2-1}{x-1} = \frac{0}{0} \ (IF)$

$\lim\limits_{x\to1} \frac{x^2-1}{x-1} = \lim\limits_{x\to1} \frac{(x-1)(x+1)}{x-1} = \lim\limits_{x\to1}(x+1) = 2$

❸ $\lim\limits_{x\to0^+} x^x = 0^0 \ (IF)$

$\lim\limits_{x\to0^+} x^x = \lim\limits_{x\to0^+} e^{\ln(x^x)} = \lim\limits_{x\to0^+} e^{x\cdot\ln(x)} = e^0 = 1$

1.2.7 Some standard limits

$\lim\limits_{x\to0} \frac{\sin x}{x} = 1, \qquad \lim\limits_{x\to0} \frac{\cos x-1}{x} = 0$

$\lim\limits_{x\to0} \frac{e^x-1}{x} = 1, \qquad \lim\limits_{x\to\infty} \frac{e^x}{x} = +\infty$

$\lim\limits_{x\to0} \frac{\ln(x+1)}{x} = 1, \qquad \lim\limits_{x\to\infty} \frac{\ln x}{x} = 0 \text{ et } \lim\limits_{x\to0^+} x\ln x = 0$

1.3 Continuous functions

Definition 1.3.1:

- Let f be a function defined on a subset $D\subseteq\mathbb{R}$ and let $x_0\in D$, we say that f is continuous at x_0 if

$$\lim_{x\to x_0} f(x) = f(x_0).$$

- We say that f is continuous on D if it is continuous at every point of D.

Checking Continuity at a Point

A function f is continuous at $x = x_0$ if the following three conditions hold:

1. $f(x_0)$ is defined (that is, x_0 belongs to the domain of f).
2. $\lim_{x \to x_0} f(x)$ exists (that is, the left-hand limit equals the right-hand limit).
3. $\lim_{x \to x_0} f(x) = f(x_0)$.

1.3.1 Operations on Continuous Functions

Let f and g be two continuous functions in x_0 at let $\lambda \in \mathbb{R}$, then
$(f+g)\,,\ (f-g)\,,\ (f \cdot g)\,,\ (\lambda f)\,,\ \left(\frac{f}{g}, (g(x_0) \neq 0)\right)\,,\ (|f|)$ are continuous functions in x_0.

1.3.2 Extension by Continuity

Let f be a function defined and continuous on a set $I \setminus \{x_0\}$, if $\lim_{x \to x_0} f(x) = \ell$ (ℓ exists and finite) then $f(x)$ can be extended by continuity at point x_0 to the function g defined by

$$g(x) = \begin{cases} f(x), & \text{if } x \neq x_0, \\ \lim_{x \to x_0} f(x), & \text{if } x = x_0. \end{cases}$$

Example 1.3.1 *The function* $f(x) = \frac{x^2-1}{x^3+1}$ *is continuous on* $\mathbb{R} - \{-1\}$ *and we have* $\lim_{x \to -1} f(x) = -\frac{2}{3}$ *therfore* $f(x)$ *can be extended by continuity at the point* $-\frac{2}{3}$ *to the function*

$$g(x) = \begin{cases} \frac{x^2-1}{x^3+1}, & \textit{if } x \neq -1, \\ -\dfrac{2}{3}, & \textit{if } x = -1. \end{cases}$$

1.4 Differentiable functions

Definition 1.4.1: Le $f : I \to \mathbb{R}$

1. We say that the function f is differentiable at the point x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

   exists and is finite.

   if $h = x - x_0$, we obtain

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

2. We say that the function f is differentiable on the right at x_0 if the limit

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0}.$$

   exists and is finite, and we denote it by $f'_+(x_0)$

3. We say that the function f is differentiable on the left at x_0 if the limit

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0}.$$

   exists and is finite, and we denote it by $f'_-(x_0)$

Remark 1.4.1 *f is differentiable at the point x_0* $\Longleftrightarrow \begin{cases} f'_+(x_0) \text{ exists and finite.} \\ f'_-(x_0) \text{ exists and finite.} \\ f'_+(x_0) = f'_-(x_0) \end{cases}$

Example 1.4.1 $f(x) = |x - 1|, \ x_0 = 1$

f is differentiable at the point $x_0 = 1$ $\Longleftrightarrow \begin{cases} f'_+(1) \text{ exists and finite.} \\ f'_-(1) \text{ exists and finite.} \\ f'_+(1) = f'_-(1) \end{cases}$

We have

$$\begin{aligned} f'_+(1) &= \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}. \\ &= \lim_{x\to 1^+} \frac{x-1}{x-1}. \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} f'_-(1) &= \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}. \\ &= \lim_{x\to 1^-} \frac{-(x-1)}{x-1}. \\ &= -1 \end{aligned}$$

we have $f'_+(1)$ *and* $f'_-(1)$ *exist but* $f'_+(1) \neq f'_-(1)$, *then the function* f *is not differentiable at the point* $x_0 = 1$.

1.4.1 Operations on differentiable functions

Let $f, g :\longrightarrow \mathbb{R}$ be two differentiable functions in x^0 and let $\lambda \in \mathbb{R}$, then

1. $(\lambda f)'(x_0) = \lambda f'(x_0)$
2. $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$
3. $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
4. If $g(x_0) \neq 0$ then
$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$$
5. **Derivative of a composite functions:** Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ such that $f(I) \subset J$, if f is differentiable at x_0 and g is differentiable at $f'(x_0)$ then $g \circ f$ is differentiable at x_0 and we have
$$(g \circ f)'(x_0) = f'(x_0)g'(f(x_0))$$

.

1.4.2 Common derivatives

Function $f(x)$	**Derivative** $f'(x)$	**Interval of Differentiability**
c	0	$\mathbb{R}$
x^n $(n \in \mathbb{N}^*)$	nx^{n-1}	$\mathbb{R}$
$\frac{1}{x}$	$-\frac{1}{x^2}$	$\mathbb{R} \setminus \{0\}$
$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$	$]0, \infty[$
e^x	e^x	$\mathbb{R}$
a^x	$a^x \ln(a)$	$\mathbb{R}$
$\ln(x)$	$\dfrac{1}{x}$	$]0, \infty[$
$\sin x$	$\cos x$	$\mathbb{R}$
$\cos x$	$-\sin x$	$\mathbb{R}$
$\tan x$	$\frac{1}{\cos^2(x)} = 1 + \tan^2(x)$	$\left]\frac{\pi}{2} + k\pi, \frac{\pi}{2} + (k+1)\pi\right[, k \in \mathbb{Z}$

Function $f(x)$	**Derivative** $f'(x)$
$[f(x)]^n$	$n\,[f(x)]^{n-1} f'(x)$
$\sqrt{f(x)}$	$\dfrac{f'(x)}{2\sqrt{f(x)}}$
$\dfrac{1}{f(x)}$	$-\dfrac{f'(x)}{[f(x)]^2}$
$\ln(f(x))$	$\dfrac{f'(x)}{f(x)}$
$e^{f(x)}$	$e^{f(x)} f'(x)$
$a^{f(x)}$	$a^{f(x)} \ln(a) f'(x)$
$\sin(f(x))$	$\cos(f(x)) f'(x)$
$\cos(f(x))$	$-\sin(f(x)) f'(x)$
$\tan(f(x))$	$\dfrac{f'(x)}{\cos^2(f(x))}$

1.4.3 Application of the derivative

1.4.3.1 Study of monotonicity

Theorem 1.4.1 *Let f be a differentiable function on $I \subset \mathbb{R}$, then*

1. *f is constant on $I \iff f'(x) = 0, \ \forall x \in I$*
2. *f is increasing on $I \iff f'(x) \geqslant 0, \ \forall x \in I$*
3. *f is strictly increasing on $I \iff f'(x) > 0, \ \forall x \in I$*
4. *f is decreasing on $I \iff f'(x) \leqslant 0, \ \forall x \in I$*
5. *f is strictly decreasing on $I \iff f'(x) < 0, \ \forall x \in I$*

1.4.3.2 Calculate the limits

Theorem 1.4.2 (Hopital's rule) *Let f, g be two defined and differentiable functions in a neighborhood v of x_0 such that $g(x) \neq 0, \ g'(x) \neq 0$ on v, if*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \ \ or \qquad \lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \pm\infty$$

Then,

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Remark 1.4.2 *Hopital's rule applies to indeterminate forms $\left(\frac{0}{0} \text{ or } \frac{\infty}{\infty}\right)$.*

Example 1.4.2

1. $\lim_{x \to +\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$*(IF), according to the Hopital's rule*

$$\lim_{x \to +\infty} \frac{x^2}{e^x} = \lim_{x \to +\infty} \frac{2x}{e^x} = \lim_{x \to +\infty} \frac{2}{e^x} = 0.$$

2. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{0}{0}$*(IF), according to the Hopital's rule*

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{2x}{1} = 2.$$

1.4.3.3 Extremum of functions

Definition 1.4.2: Let $f : I \longrightarrow \mathbb{R}$ and let $x_0 \in I$

❶We say that f admits a local maximum at x_0 if

$$\exists J \subset I \text{ of center } x_0, \forall x \in \quad J f(x) \leq f(x_0)$$

❷We say that f admits a local minimum at x_0 if

$$\exists J \subset I \text{ of center } x_0, \forall x \in \quad J f(x) \geq f(x_0)$$

❸We say that f admits a local extremum at x_0 if f admits at x_0 a local maximum or minimum.

Critical point

Definition 1.4.3: A critical point of a function f is a point c in its domain for which $f'(c) = 0$ or $f'(c)$ does not exist.

Corollary 1.4.1 *If $f'(c)$ exists and $f'(c) \neq 0$, then $f(c)$ is not a local extremum of the function f.*

Theorem 1.4.3 *Let $I \subseteq \mathbb{R}$, let $c \in I$ and let $f : I \longrightarrow \mathbb{R}$ be a function. Suppose that f is differentiable, that $f'(c) = 0$ and f is twice differentiable at c*

❶*If $f''(c) > 0$, then c is a local minimum of f.*

❷*If $f''(c) < 0$, then c is a local maximum of f.*

Example 1.4.3 *Find the extremum of the following functions*

❶$f(x) = x^2 - 1$

The critical points

We have

$$f'(x) = 2x$$

So

$$f'(x) = 0 \Longleftrightarrow 2x = 0 \Longleftrightarrow x = 0 \, is \; a \; critical \; point \; off$$

The nature of critical points

$f''(x) = 2 \neq 0$ therefore f admits a minimum at the point $x_0 = 0$ because $f''(0) > 0$

❷$f(x) = x^2e^{-x}$

The critical points

We have

$$f'(x) = xe^{-x}(2-x)$$

So

$$f'(x) = 0 \Longleftrightarrow xe^{-x}(2-x) = 0 \Longleftrightarrow x = 0 \ or \ \ x = 2$$

The nature of critical points

We have

$$f''(x) = e^{-x}(1-x)(2-x) - xe^{-x}$$

for $x = 0, \ f''(0) = 2 > 0$ *so* $x = 0$ *is a minimum.*

for $x = 2, \ f''(0) = -2e^{-2} < 0$ *so* $x = 2$ *is a maximum.*