Tutorial Sheet — Comparison of Means (Solved Exercises)

Level: L3

Exercise 1 — Two-Sample t-Test (Independent Samples)

Problem. Two teaching methods are tested. The final exam scores (out of 100) of students are:

- Method 1: 85, 88, 90, 87, 86
- Method 2: 80, 82, 78, 81, 79

Test whether there is a significant difference between the two population means at significance level $\alpha = 0.05$. Assume equal variances.

Solution.
Step 1: Hypotheses

$$\begin{aligned} H_0 &: \mu_1 = \mu_2 \quad \text{(no difference in means)}, \\ H_1 &: \mu_1 \neq \mu_2 \quad \text{(means are different)}. \end{aligned}$$

Step 2: Sample statistics
Each sample size $n_1 = n_2 = 5$.
Group means:

$$\bar{x}_1 = \frac{85 + 88 + 90 + 87 + 86}{5} = 87.2, \qquad \bar{x}_2 = \frac{80 + 82 + 78 + 81 + 79}{5} = 80.0.$$

Sample variances (unbiased):

$$s_1^2 = \frac{\sum (x_{1i} - \bar{x}_1)^2}{n_1 - 1} = 3.70, \qquad s_2^2 = 2.50.$$

Pooled variance:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{4 \cdot 3.70 + 4 \cdot 2.50}{8} = 3.10.$$

Step 3: Test statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}} = \frac{87.2 - 80.0}{\sqrt{3.10 \left(\frac{1}{5} + \frac{1}{5} \right)}} = \frac{7.2}{\sqrt{3.10 \times 0.4}} \approx 6.47.$$

Degrees of freedom: $df = n_1 + n_2 - 2 = 8$.
Step 4: Decision Two-sided critical value $t_{0.025,8} \approx 2.306$. Since $|t| = 6.47 > 2.306$, reject H_0.
Conclusion: There is strong evidence (at the 5% level) that the two teaching methods have different mean scores. Method 1 yields significantly higher mean.

Exercise 2 — Paired (Dependent) Samples t-Test

Problem. A training program is tested on the same group of students: the exam score is recorded before and after the program. The scores are:

- Before: 70, 72, 68, 75, 74, 71
- After: 74, 75, 70, 78, 77, 74

Test whether the program produces a significant change in scores at $\alpha = 0.05$.

Solution.
Step 1: Hypotheses

$$H_0 : \mu_d = 0 \quad \text{(no mean difference)},$$
$$H_1 : \mu_d \neq 0 \quad \text{(mean difference exists)},$$

where $d = $ (After Before).
Step 2: Differences Compute individual differences d_i:

$$d = [4,\ 3,\ 2,\ 3,\ 3,\ 3].$$

Sample size $n = 6$.
Mean of differences:

$$\bar{d} = \frac{4+3+2+3+3+3}{6} = \frac{18}{6} = 3.0.$$

Sample variance of differences: first compute squared deviations from $\bar{d}$:

$$(4-3)^2 = 1,\ (3-3)^2 = 0,\ (2-3)^2 = 1,\ (3-3)^2 = 0,\ (3-3)^2 = 0,\ (3-3)^2 = 0.$$

Sum $=$ 2. So

$$s_d^2 = \frac{2}{n-1} = \frac{2}{5} = 0.4, \qquad s_d = \sqrt{0.4} \approx 0.6325.$$

Step 3: Test statistic

$$t = \frac{\bar{d} - 0}{s_d/\sqrt{n}} = \frac{3.0}{0.6325/\sqrt{6}} = \frac{3.0}{0.2582} \approx 11.62.$$

Degrees of freedom: $df = n - 1 = 5$.
Step 4: Decision Two-sided critical value $t_{0.025,5} \approx 2.571$. Since $11.62 > 2.571$, reject H_0.
Conclusion: The training program produces a statistically significant increase in scores (very strong evidence).

Remark: For paired tests, always check that data are paired and differences are approximately normal (especially for small n).

Exercise 3 — One-Way ANOVA (Three Fertilizers)

Problem. A researcher tests whether three types of fertilizer (A, B and C) have different effects on plant growth. After 4 weeks:

Fertilizer	Heights (cm)			
A	20	21	19	22
B	25	27	26	28
C	23	22	24	21

Test at significance level $\alpha = 0.05$ whether there is a difference between the mean heights.

Solution.
Step 1: Hypotheses

$$H_0 : \mu_A = \mu_B = \mu_C,$$
$$H_1 : \text{At least one mean differs.}$$

Step 2: Sample sizes and means
Each group has $n = 4$, total $N = 12$.
Group means:

$$\bar{X}_A = \frac{20 + 21 + 19 + 22}{4} = 20.5, \quad \bar{X}_B = 26.5, \quad \bar{X}_C = 22.5.$$

Grand mean:

$$\bar{X} = \frac{20 + 21 + 19 + 22 + 25 + 27 + 26 + 28 + 23 + 22 + 24 + 21}{12} = \frac{278}{12} \approx 23.1667.$$

Step 3: Sum of squares
Between-groups sum of squares (SSB):

$$\begin{aligned} SSB &= n\sum_{i=1}^{3}(\bar{X}_i - \bar{X})^2 \\ &= 4\big[(20.5 - 23.1667)^2 + (26.5 - 23.1667)^2 + (22.5 - 23.1667)^2\big] \\ &= 4(7.1111 + 11.1111 + 0.4444) = 74.6667. \end{aligned}$$

Within-groups sum of squares (SSW) — sum of squared deviations within each group:

$$\begin{aligned} SSW_A &= (20 - 20.5)^2 + (21 - 20.5)^2 + (19 - 20.5)^2 + (22 - 20.5)^2 = 5, \\ SSW_B &= (25 - 26.5)^2 + (27 - 26.5)^2 + (26 - 26.5)^2 + (28 - 26.5)^2 = 5, \\ SSW_C &= (23 - 22.5)^2 + (22 - 22.5)^2 + (24 - 22.5)^2 + (21 - 22.5)^2 = 5. \end{aligned}$$

So

$$SSW = 5 + 5 + 5 = 15.$$

Total sum of squares check:

$$SST = SSB + SSW = 74.6667 + 15 = 89.6667.$$

Step 4: ANOVA table and F statistic
Degrees of freedom:

$$df_{between} = k - 1 = 3 - 1 = 2, \qquad df_{within} = N - k = 12 - 3 = 9.$$

Mean squares:

$$MSB = \frac{SSB}{df_{between}} = \frac{74.6667}{2} = 37.3333,$$

$$MSE = \frac{SSW}{df_{within}} = \frac{15}{9} = 1.6667.$$

F statistic:

$$F = \frac{MSB}{MSE} = \frac{37.3333}{1.6667} = 22.4.$$

Source	SS	df	MS	F
$Betweengroups$	74.6667	2	37.3333	22.4
$Withingroups$	15.0000	9	1.6667	
$Total$	89.6667	11		

Step 5: Decision At $\alpha = 0.05$ with $(df_1, df_2) = (2, 9)$, critical value $F_{0.05;2,9} \approx 4.26$. Since $F = 22.4 > 4.26$, reject H_0.

Conclusion: There is significant evidence at the 5% level that at least one fertilizer produces a different mean plant height. Use a post-hoc test (e.g. Tukey HSD) to identify which pairs differ.

Prepared by: Azioune Samir