

Solution of Series of Exercises 2

Solution of Exercise 1

Let us study the simple convergence and uniform convergence of the following sequences of functions:

1. $f_n(x) = \frac{1 - nx^2}{1 + nx^2}$, $E_1 = \mathbb{R}$ then on $E_2 = [a, +\infty[$ ($a > 0$).

1.1 Simple Convergence :

If $x = 0$, $f_n(0) = 1$ and if $x \neq 0$, $\lim f_n(x) = -1$. the sequence of fonctions (f_n) is therefore simply convergente on $E_1 = [-a, a]$ towards f , where $f(x) = \begin{cases} 1, & \text{if } x = 0, \\ -1, & \text{if } x \neq 0 \end{cases}$.

1.2 Uniform Convergence :

1.2.1. On $E_1 = \mathbb{R}$. The function f is not continuous on $\mathbb{R}$, it folloxs that (f_n) doesn't converge uniformly on E_1 .

1.2.2. On $E_2 = [a, +\infty[$ ($a > 0$). We have

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{1 - nx^2}{1 + nx^2} - (-1) \right| \\ &= \left| \frac{2}{1 + nx^2} \right| \\ &\leq \frac{2}{1 + na^2}. \end{aligned}$$

This later quantity tends towards 0, and the sequence of functions converges uniformly to -1 on $E_2 = [a, +\infty[$ ($a > 0$).

2. $f_n(x) = \frac{x}{1 + nx}$, $E_3 = [0, 1]$.

2.1 Simple Convergence :

If $x = 0$, $f_n(0) = 0$ and if $x \neq 0$, $\lim f_n(x) = 0$.

The sequence of functions (f_n) is therefore simply convergent on $E_3 = [0, 1]$ to the zero function $f = 0$.

2.2 Uniform Convergence :

Since each function $x \mapsto f_n(x) - f(x) = \frac{x}{1 + nx}$, is increasing on E_3 , we then have $\sup |f_n(x) - f(x)| = \frac{1}{1 + n}$. This later quantity tends to 0, and the sequence of functions converges uniformly to 0 on E_3 .

Solution of Exercise 1

3. $f_n(x) = \cos(\frac{5+nx}{n})$, $E_4 = \mathbb{R}$.

3.1 Simple Convergence :

If $x = 0$, $f_n(0) = \cos(\frac{5}{n})$, and $\lim f_n(0) = 1$.

When $x \neq 0$, $\lim f_n(x) = \cos(x)$.

In all cases, the sequence of functions (f_n) is therefore simply convergent on $E_4 = \mathbb{R}$ to f , where $f(x) = \cos(x)$.

3.2 Uniform Convergence :

We have

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \cos(\frac{5+nx}{n}) - \cos(x) \right| \\ &= \left| \cos(x) \times \cos(\frac{5}{n}) - \sin(x) \times \sin(\frac{5}{n}) - \cos(x) \right| \\ &\leq |\cos(x)| \times \left| \cos(\frac{5}{n}) - 1 \right| + |\sin(x)| \times \left| \sin(\frac{5}{n}) \right| \\ &\leq \frac{25}{2n^2} + \frac{5}{n}. \end{aligned}$$

This later quantity tends to 0, and the sequence of functions converges uniformly to 0 on E_4 .

4. $f_n(x) = \frac{\sin(nx)}{nx}$ and $f_n(0) = 0$, $E_5 = \mathbb{R}$ then on $E_6 = [a, +\infty[$ ($a > 0$)

4.1 Simple Convergence :

If $x = 0$, $f_n(0) = 0 \Rightarrow \lim f_n(0) = 0$, and if $x \neq 0$, $\lim f_n(x) = 0$. the sequence of functions (f_n) is therefore simply convergent on $\mathbb{R}$ to $f(x) = 0$.

4.2 Uniform Convergence :

4.2.1 On $E_5 = \mathbb{R}$. Let's consider the sequence of points $(x_n = \frac{\pi}{2n})$. We have

$|f_n(x_n) - f(x_n)| = \frac{2}{\pi} \not\rightarrow 0$. It follows that (f_n) doesn't converge uniformly on $E_5 = \mathbb{R}$.

4.2.2. On $E_6 = [a, +\infty[$ ($a > 0$). We have

$$|f_n(x) - f(x)| = \left| \frac{\sin(nx)}{nx} \right| \leq \frac{1}{nx} \leq \frac{1}{na}.$$

This later quantity tends to 0, and the sequence of functions converges uniformly to 0 on $E_6 = [a, +\infty[$ ($a > 0$).

Solution of Exercise 1

5. $f_n(x) = x^n(1 - x)$, $E_7 = [0, 1]$.

5.1 Simple Convergence :

If $x = 0$ or $x = 1$, $f_n(0) = f_n(1) = 0$, and the sequence converges to 0. If $x \in]0, 1[$, $\lim_{n \rightarrow +\infty} f_n(x) = 0$.

Finally, the sequence (f_n) converges simply on $[0, 1]$ and this limit is the zero function $f(x) = 0$.

5.2 Uniform Convergence :

By performing a simple calculation, we find

$$\sup_{x \in [0, 1]} |f_n(x) - f(x)| = a_n = \frac{1}{n+1} \left(\frac{n}{n+1} \right)^n.$$

This latter quantity equivalent in neighborhood of infinity by $\frac{1}{ne}$.

Since this quantity tends to 0, when n tends to $+\infty$, therefore the considered sequence converges uniformly to 0 on $[0, 1]$.

Solution of Exercise 2

Let us consider the series of functions:

$$f_n(x) = \sin^2(x) \cos^n(x), \text{ for } n \geq 0 \text{ and } x \in \left[0, \frac{\pi}{2}\right].$$

1. If $x = 0$, $\sum_{n \geq 0} f_n(x) = 0$ and the series converges.

If $x \in \left]0, \frac{\pi}{2}\right]$, the series $\sum_{n \geq 0} f_n$, is geometric with ratio $q = \cos(x) \in [0, 1[$, and therefore it converges on $\left]0, \frac{\pi}{2}\right]$.

Finally:

$$S(x) = \begin{cases} \frac{\sin^2(x)}{1 - \cos(x)}, & \text{if } x \in \left]0, \frac{\pi}{2}\right] \\ 0, & \text{if } x = 0. \end{cases} \quad (1)$$

2. Since S is not continuous on the segment $\left[0, \frac{\pi}{2}\right]$, the series $\sum_{n \geq 0} f_n$, is therefore not uniformly convergent on this segment..

Solution of Exercise 3

Let us consider the series of functions:

$$f_n(x) = \frac{x}{(1+x^2)^n}, \text{ for } n \geq 1 \text{ and } x \in \mathbb{R},$$

1. If $x = 0$, $\sum_{n \geq 1} f_n(x) = 0$, the series is simply convergent.

If $x \neq 0$, the series $\sum_{n \geq 1} f_n(x)$ is geometric series with ratio $q = \frac{1}{1+x^2} < 1$, and therefore it converges for all $x \neq 0$.

Finally :

$$S(x) = \begin{cases} \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases} \quad (2)$$

2. Since S is not continuous at the point 0, the series $\sum_{n \geq 1} f_n$, is therefore not uniformly convergent on $\mathbb{R}$.

3. Let's study the normal convergence on $[a, b]$ ($0 < a < b$). We have

$$\begin{aligned} |f_n(x)| &= \frac{x}{(1+x^2)^n} \\ &\leq \frac{x}{(1+a^2)^n} \\ &\leq \frac{b}{(1+a^2)^n}, \forall x \in [a, b], \end{aligned}$$

which is the general term of convergent geometric series, and so $\sum_{n \geq 1} f_n$ converges uniformly on $[a, b]$.

4. Calculate $\sum_{n \geq 1} \int_1^e f_n(x) dx$:

Since the series is uniformly convergent on $[1, e]$, it is therefore integrable term by term on this segment. We then have

$$\begin{aligned} \sum_{n \geq 1} \int_1^e f_n(x) dx &= \int_1^e \sum_{n \geq 1} f_n(x) dx \\ &= \int_1^e \frac{dx}{x} \\ &= 1. \end{aligned}$$