Series of Exercises 2

Exercise 1

Let a be a positive real number and let (f_n) be a sequence of functions defined on the set $E_i \in \mathbb{R}$. Study the simple and uniform convergence of this sequence of functions on E_i in the following cases:

❶ $f_n(x) = \dfrac{1 - nx^2}{1 + nx^2}$, $E_1 = \mathbb{R}$, then on $E_2 = [a, +\infty[$.

❷ $f_n(x) = \dfrac{x}{1 + nx}$, $E_3 = [0, 1]$.

❸ $f_n(x) = \cos(\dfrac{5 + nx}{n})$, $E_4 = \mathbb{R}$.

❹ $f_n(x) = \dfrac{\sin(nx)}{nx}$, $f_n(0) = 0$, $E_5 = \mathbb{R}$, then on $E_6 = [a, +\infty[$.

❺ $f_n(x) = x^n(1 - x)$, $E_7 = [0, 1]$.

Exercise 2

Consider the series of functions with general term:

$$f_n(x) = \sin^2(x)\cos^n(x), \text{ for } n \geq 0 \text{ and } x \in \left[0, \frac{\pi}{2}\right].$$

❶ Prove that the series of functions $\sum f_n(x)$ converges simply on $\left[0, \dfrac{\pi}{2}\right]$ and calculate its sum.

❷ Is the series uniformly convergent on $\left[0, \dfrac{\pi}{2}\right]$?

Exercise 3

Consider the series of functions with general term:

$$f_n(x) = \frac{x}{(1+x^2)^n}, \ n \geq 1 \text{ and } x \in \mathbb{R}.$$

❶ Prove that the series of functions $\sum f_n(x)$ converges simply on $\mathbb{R}$ and calculate its sum.

❷ Is the series uniformly convergent on $\mathbb{R}$?

❸ Study the normal convergence on $[a, b]$, then on $[a, +\infty[$, $(0 < a < b)$.

❹ Calculate $\sum_{n\geq 1} \int_1^e f_n(x)dx$.