Mathematics 3

Chapter 1: Simple and multiple integrals

1.1 Riemann sums and definite integrals.

1.2 Double and triple integrals.

1.3 Applications of integration:area , volumes.

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CHAPTER 1

SIMPLE AND MULTIPLE INTEGRALS

1.1 Riemann integral and fundamental theorem of calculus.

1.1.1 Review

Definition 1.1 *A partition of an interval $[a,b]$ is a finite sequence of numbers of the form*

$$a = x_0 < x_1 < x_3 < ... < x_{n-1} < x_n = b.$$

Each $[x_i, x_{i+1}]$ is called a sub-interval of the partition.
The mesh or norm of a partition is defined to be the lenght of the largest sub-interval, that is $\max(x_{i+1} - x_i)$, $i \in [0, n-1]$.

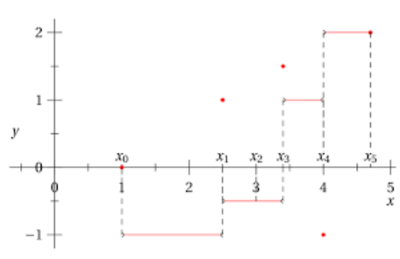
Definition 1.2 *A function $f : [a,b] \to \mathbb{R}$ is called a step function if it is piecewise constant, i.e if there are numbers*

$$a = x_0 < x_1 < x_3 < ... < x_{n-1} < x_n = b$$

such that f is constant on each half open interval $[x_{i-1}, x_i)$ with $1 \leq i \leq n$. For a step function we define the integral to be

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

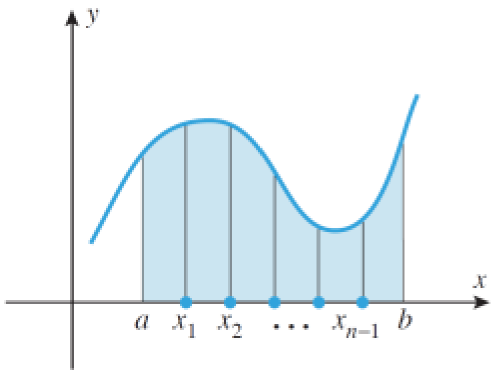
The collection of numbers $\{x_0, x_1, ..., x_n\}$ are called a partition for the step function.



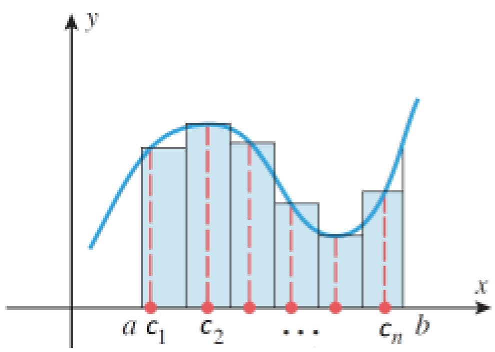
1.1.2 Riemann integral

1. A partition of $[a, b]$: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, $x_k = \dfrac{b-a}{n}k + a$, $k = 0, 1, \ldots, n$ divides $[a, b]$ into n subintervals $[a_{k-1}, a_k]$ with width:

$$\Delta x_k = x_k - x_{k-1} = \frac{b-a}{n}, \quad k = 1, 2, \ldots, n.$$



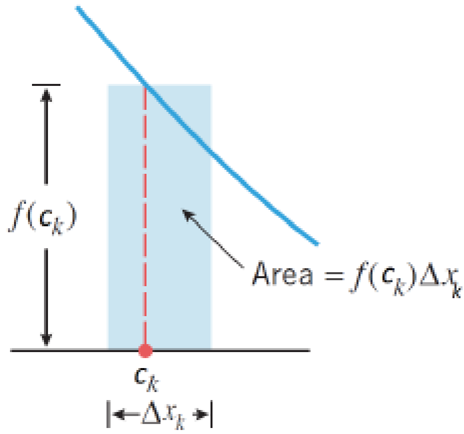
2. Choose points $c_k \in [x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, to form small rectangles.



3. Calculate the area of each rectangle and sum them up.

   For the kth subinterval,

$$\boxed{\text{Area of } k\text{th rectangle} = \text{height} \times \text{width} = f(c_k)\Delta x_k}$$



Definition 1.3

$$S_n = \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right)$$

is called a Rieman sum of f on $[a,b]$.

if f is Riemann integrable(continous or monotone). Then we have

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n \longrightarrow +\infty} S_n \\ &= \lim_{n \longrightarrow +\infty} \sum_{i=1}^{n} \frac{b-a}{n} f\left(a + i\frac{b-a}{n}\right)\end{aligned}$$

Example 1 *Evaluate $\int_0^1 e^x dx$ using the Riemann sum.*

1.1.3 Fundamental theorem of calculus

Definition 1.4 *Let $f : I \longrightarrow \mathbb{R}$ be a function. We say that $F : I \longrightarrow \mathbb{R}$ is antiderivative (primitive) of f on I if F is a derivable function on I satisfies $F'(x) = f(x)$ for all $x \in I$.*

Example 2 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by* $f(x) = x^2$.*Then*

$$\begin{aligned} F: \quad \mathbb{R} &\longrightarrow \mathbb{R} \\ x :&\longrightarrow \tfrac{x^3}{3} \end{aligned}$$

is an antiderivative of f.
The function $F(x) = \frac{x^3}{3} + 9$ *is also an antiderivative of* f.

Proposition 1.1 *Let* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *and* $F : \mathbb{R} \longrightarrow \mathbb{R}$ *be a primitive of* f . *For all primitive of* f *is written as* $G = F + c$ *where* $c \in \mathbb{R}$.

Theorem 1.1.1 *Let* f *be a continous function on* $[a, b]$. *For all primitive F of* f *we have:*

$$\int_a^b f(x)\, dx = [F(x)]_a^b = F(b) - F(a).$$

Example 3 *Evaluate the following integral.*

$$\int_0^1 \left(5x^4 + e^{2x} - \frac{1}{1+x}\right) dx$$

Primitives of usual functions

Function f	primitive F	Interval
x^α, $\alpha \neq -1$	$\frac{x^{\alpha+1}}{\alpha+1}$	$\mathbb{R}^+$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$]-\infty,0[$ ou $]0,+\infty[$
e^x	e^x	$\mathbb{R}$
$\cos x$	$\sin x$	$\mathbb{R}$
$\sin x$	$-\cos x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x$	$]-1,1[$
$\frac{1}{1+x^2}$	$\arctan x$	$\mathbb{R}$

Let g be a continous function on $[a,b]$.

Function f	primitive F
$g'(x)g^\alpha(x)$, $\alpha \neq -1$	$\frac{g^{\alpha+1}(x)}{\alpha+1}$
$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$
$g'(x)e^{g(x)}$	$e^{g(x)}$
$g'(x)\cos[g(x)]$	$\sin[g(x)]$
$g'(x)\sin[g(x)]$	$-\cos[g(x)]$
$\frac{g'(x)}{\sqrt{1-g^2(x)}}$	$\arcsin[g(x)]$
$\frac{g'(x)}{1+g^2(x)}$	$\arctan[g(x)]$

Properties of definite integrals

Let f and g be two continous functions on $[a,b]$. We have

1) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$

2) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx;\ c \in [a,b].$

3) $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx;\ \lambda \in \mathbb{R}.$

4) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$

5) $\int_a^a f(x)\,dx = 0.$

Techniques of integration

Integration by Parts

Let u and v be two continous functions on $[a,b]$. Then

$$\int_a^b uv'dx = [uv]_a^b - \int_a^b u'vdx.$$

Example 4 *Evaluate* $\int_0^1 xe^x dx$

Example 5 *Evaluate* $\int arcsinxdx$

Integration by Substitution(Change of variables)

Let f be a continous function on $[a,b]$ and u be a derivable function. Then if we suppose $x = u(t)$ we obtain the following

$$\int_a^b f(x)\,dx = \int_{u^{-1}(a)}^{u^{-1}(b)} f[u(t)]\,u'(t)\,dt.$$

Example 6 *Find* $\int_0^{ln2} \sqrt{e^x - 1}dx$

Example 7 *Evaluate* $\int tanxdx$

Integration of rational functions

Now, we consider the general problem of evaluating , $\int \frac{P(x)}{Q(x)}dx$ where $P(x)$ and $Q(x)$ are polynomials of x.

1-**Evaluation of** $\int \frac{P(x)}{Q(x)}dx$ **when** $deg(P(x)) < deg(Q(x))$ **(proper rational function)**

a)If $Q(x) = 0$ has simple real roots

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + ... + \frac{A_n}{x - x_n}, \text{ such that } A_1, A_2, ..., A_n \in \mathbb{R}$$

where $\int \frac{1}{x - x_i}dx = ln|x - x_i| + c$, $i = 1, 2, ..., n$.

b)If $Q(x) = 0$ has multiple roots

$$\frac{P(x)}{Q(x)} = \frac{A_{11}}{x - x_1} + \frac{A_{12}}{(x - x_1)^2} + ... + \frac{A_{nm_n}}{(x - x_{nm_n})^{m_n}}, \text{ such that } A_{11}, A_{12}, ..., A_{nm_n} \in \mathbb{R}$$

m_i is the degree of multiplicity of the root x_i

where $\begin{cases} \int \frac{1}{x-a}dx = ln|x - a| + c \\ \int \frac{-1}{(x-a)^m}dx = \frac{1}{m-1}\frac{1}{(x-a)^{m-1}} + c. \end{cases}$

c)If $Q(x) = x^2 + px + q$ has complexe roots. Then $Q(x) = \left(x + \frac{p}{2}\right)^2 - \frac{p^2}{4} + q$.

2-**Evaluation of** $\int \frac{P(x)}{Q(x)}dx$ **when** $deg(P(x)) > deg(Q(x))$ **(improper rational function)**

We apply the Euclidean division of $P(x)$ by $Q(x)$

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where S is a polynomial and the the degree of remainder $R(x)$ less than the degree of $Q(x)$. Then

$$\int \frac{P(x)}{Q(x)}dx = \int S(x)\,dx + \int \frac{R(x)}{Q(x)}dx.$$

Example 8 *Evaluate the following integral* $\int \frac{x}{(x+1)(2x+1)}dx$.

Example 9 *Find* $\int \frac{x^5+x^4-8}{x^3-4x}dx$

Integration of powers of trigonometric functions

Consider $I = \int sin^n x cos^m x dx$ where $n, m \in \mathbb{N}$.

- If n is odd, use $t = cosx$
- If m is odd, use $t = sinx$
- If n and m are odd, use either $t = cosx$ or $t = sinx$
- If n and m are even, use $cos^2 x = \frac{1}{2}(1 + cos(2x))$, and/or $sin^2 x = \frac{1}{2}(1 - cos(2x))$ to reduce to a form that can be integrated.

Example 10 *Evaluate* $\int sin^4 x cos^5 x dx$

Integration of rational function of sine and cosine

Let $I = \int R(sinx, cosx)\, dx$ where R is rational function of $sinx$ and $cosx$

To evaluate I we make the substitution $t = tan\frac{x}{2}$, so $x = 2arctant$ and $dx = \frac{2}{1+t^2}dt$

$$sinx = \frac{2sin\left(\frac{x}{2}\right)cos\left(\frac{x}{2}\right)}{cos^2\left(\frac{x}{2}\right) + sin^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2}$$

$$cosx = \frac{cos^2\left(\frac{x}{2}\right) - sin^2\left(\frac{x}{2}\right)}{cos^2\left(\frac{x}{2}\right) + sin^2\left(\frac{x}{2}\right)} = \frac{1-t^2}{1+t^2}$$

$$\begin{aligned}\cos 2a &= \cos^2 a - \sin^2 a \\ &= 2\cos^2 a - 1 \\ &= 1 - 2\sin^2 a \\ &= \frac{1-\tan^2 a}{1+\tan^2 a} \\ \sin 2a &= 2\sin a \cos a \\ \tan 2a &= \frac{2\tan a}{1-\tan^2 a}\end{aligned}$$

Example 11 $I = \int \frac{1}{sinx}dx$

Example 12 $J = \int \frac{dx}{cosx}$

Integraion of the forme $I = \int R\left(x, \sqrt{ax^2+bx+c}\right)dx$
where R is rational function of $\sqrt{ax^2+bx+c}$
$*$ We write $ax^2+bx+c = a\left(\left(x+\frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right)$
$*$ By using the change of variables, we get one of the following $\sqrt{1+t^2}$, $\sqrt{t^2-1}$, $\sqrt{1-t^2}$
Let $t = shu$ for $\sqrt{1+t^2} \Rightarrow dt = chudu$
$t = chu$ for $\sqrt{t^2-1} \Rightarrow dt = shudu$
$t = sinu$ for $\sqrt{1-t^2} \Rightarrow dt = cosudu.$

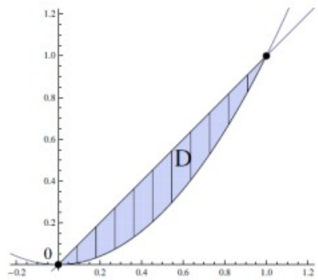
Example 13 $I = \int \sqrt{x^2+4x+2}dx = 2\int \sqrt{\left(\frac{x+2}{\sqrt{2}}\right)^2 - 1}dx$

1.2 Double and triple integrals

1.2.1 Double integrals

Let $f : D \longrightarrow \mathbb{R}$ be a function defined on $D \subset \mathbb{R}^2$. The double integral of f over $D \subset \mathbb{R}^2$ is written as

$$\int\int_D f(x,y)\,dxdy.$$



Properties of double integrals

1) The double integral over a domain D is linear.

$$\int\int_D (\alpha f + \beta g)(x,y)\,dxdy = \alpha \int\int_D f(x,y)\,dxdy + \beta \int\int_D g(x,y)\,dxdy.$$

2) If D and D' are two domains such that $D \bigcap D' = \{\varnothing$, or a courbe or isolate points $\}$

$$\int\int_{D\bigcap D'} f(x,y)\,dxdy = \int\int_D f(x,y)\,dxdy + \int\int_{D'} f(x,y)\,dxdy.$$

3) If $f(x,y) \geqslant 0$ at every point in D with f is not identically zero, then $\int\int_D f(x,y)\,dxdy$ is strictly positive.

4) If $\forall (x,y) \in D$, $f(x,y) \leqslant g(x,y) \Rightarrow \int\int_D f(x,y)\,dxdy \leqslant \int\int_D g(x,y)\,dxdy$.

5) $\left|\int\int_D f(x,y)\,dxdy\right| \leqslant \int\int_D |f(x,y)|\,dxdy$.

Fubini's theorem

Theorem 1.2.1 *Let $f : D \longrightarrow \mathbb{R}$ be a continous function on a rectangle $D = [a,b] \times [c,d]$. We have*

$$\int\int_D f(x,y)\,dxdy = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy.$$

Example 14 *Calculate* $I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} sin\,(x+y)\,dxdy$

Example 15 *Evaluate* $I = \int_0^1 \int_2^5 \frac{1}{(1+x+2y)^2} dxdy$

Speacial case

If $g : [a,b] \to \mathbb{R}$ and $h : [c,d] \to \mathbb{R}$ are two functions, then

$$\int_a^b \int_c^d g\,(x)\,h\,(y)\,dxdy = \left(\int_a^b g\,(x)\,dx\right)\left(\int_c^d h\,(y)\,dy\right).$$

Example 16 *Clculate* $I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} sinxcosydxdy$

$$\begin{aligned} I = & \left(\int_0^{\frac{\pi}{2}} sinxdx\right)\left(\int_0^{\frac{\pi}{2}} cosydy\right) \\ = & [-cosx]_0^{\frac{\pi}{2}} [siny]_0^{\frac{\pi}{2}} \\ = & 1. \end{aligned}$$

Theorem 1.2.2 *Let f be a continous function on a bounded domain $D \subset \mathbb{R}^2$. The double integral $\int\int_D f(x,y)\,dxdy$ is calculated as follow*
$*$If we can represent the domain D as the form $D = \left\{(x,y) \in \mathbb{R}^2 : f_1(x) \leq y \leq f_2(x), a \leq x \leq b\right\}$, then $\int\int_D f(x,y)\,dxdy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy\right] dx$.
$*$If we can represent the domain D as the form $D = \left\{(x,y) \in \mathbb{R}^2 : g_1(y) \leq x \leq g_2(y), c \leq y \leq d\right\}$, then $\int\int_D f(x,y)\,dxdy = \int_c^d \left[\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\right] dy$.
If the two representations are possible, then the resultes are equal.

Example 17 *Calculate the integral $\int\int_D \left(x^2 + y^2\right) dxdy$ where D is the rectangle of sommets $(0,1)$, $(0,-1)$ and $(1,0)$.*

Example 18 *Calculate the following double integral* $\int\int_D xe^y dxdy$

where $D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, x^2 \leq y \leq x\}$.

Example 19 *Calculate the following double integral* $\int\int_D \frac{dxdy}{(x+y)^3}$
where $D = \{(x,y) \in \mathbb{R}^2 : x > 1, y > 1, x + y < 3\}$.

Change of variables in double integrals

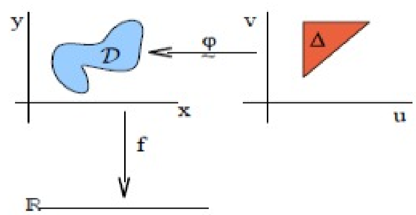
Definition 1.5 *Let $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a differentiable map. The Jacobbian matrix of φ is given by $\frac{\partial \varphi_i}{\partial x_j}$, $i,j = 1,2,...,n$, that is*

$$\nabla \varphi = \begin{bmatrix} \frac{\partial \varphi_1}{\partial x_1} & ... & ... & \frac{\partial \varphi_1}{\partial x_n} \\ \frac{\partial \varphi_2}{\partial x_1} & ... & ... & \frac{\partial \varphi_2}{\partial x_n} \\ ... & ... & ... & ... \\ \frac{\partial \varphi_n}{\partial x_1} & ... & ... & \frac{\partial \varphi_n}{\partial x_n} \end{bmatrix}$$

The determinant for the above Jacobian matrix is called a Jocobian. It will be denoted by J_φ

Theorem 1.2.3 *Let $(u,v) \in \Delta \to (x,y) = \varphi(u,v) \in D$ be a $\mathcal{C}^1$-diffeomorphism from Δ to D. For any continous function f in D.*

$$\int\int_D f(x,y)\,dxdy = \int\int_\Delta f \circ \varphi(u,v)\,|J_\varphi(u,v)|\,dudv.$$



Example 20 *Calculate $I = \int\int_D (x-1)^2\,dxdy$ over the domain*
$D = \{(x,y) \in \mathbb{R}^2 : -1 \leq x+y \leq 1, -2 \leq x-y \leq 2\}$

Change of variables in polar coordinates

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(r,\theta) \to (rcos\theta, rsin\theta)$. Then φ is $\mathcal{C}^1$-diffeomorphism on $\mathbb{R}^2$, and its Jacobian is equal to

$$J_\varphi = (r,\theta) = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$$

Then $\int\int_D f(x,y)\,dxdy = \int\int_\Delta g(r,\theta)\,rdrd\theta$.

Example 21 *Find* $I = \int\int_D \sqrt{x^2+y^2}dxdy$
where $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \geq 4, x^2+y^2 \leq 9\}$

Example 22 *Evaluate* $I = \int\int_D \frac{1}{x^2+y^2}dxdy$
where $D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2+y^2 \leq 4, x \geq 0, y \geq 0\}$

1.2.2 Triple integrals

Let $f : D \to \mathbb{R}$ be a definite function from a domain D to $\mathbb{R}^3$. Then the integral of f over D is said the triple integral, and it is denoted by $\int\int\int_D f(x,y,z)\,dxdydz$.

Fubini's Theorem

Theorem 1.2.4 *Let $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a continous function*
• *If D is a parallelepiped, then $D = [a,b] \times [c,d] \times [e,f]$*

$$\int\int\int_D f(x,y,z)\,dxdydz = \int_a^b \left(\int_c^d \left(\int_e^f f(x,y,z)\,dz\right) dy\right) dx.$$

• *If D is any bounded set, then*
$D = \left\{(x,y,z) \in \mathbb{R}^3 : x \in [a,b], y \in [f_1(x), f_2(x)], z \in [g_1(x,y), g_2(x,y)]\right\}$

$$\int\int\int_D f(x,y,z)\,dxdydz = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\right) dy\right) dx.$$

Example 23 *Calculate the following integral $I = \int\int\int_D x^2yzdxdydz$, and $D = [0,1] \times [1,2] \times [0,2]$.*

$$\begin{aligned}
I &= \int_0^2 \int_1^2 \int_0^1 x^2yzdxdydz \\
&= \left(\int_0^1 x^2dx\right)\left(\int_1^2 ydy\right)\left(\int_0^2 zdz\right) \\
&= \left[\frac{x^3}{3}\right]_0^1 \left[\frac{y^2}{2}\right]_1^2 \left[\frac{z^2}{2}\right]_0^2 \\
&= 1.
\end{aligned}$$

Example 24 *Calculate the following triple integral: $I = \int\int\int_D \frac{1}{(1+x+y+z)^3}dxdydz$*

and $D = \left\{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1\right\}$

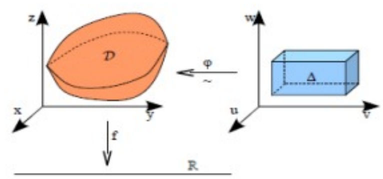
Example 25 *Find* $I = \int\int\int_D z dxdydz$
where $D = \left\{(x,y,z) \in \mathbb{R}^3, x \geq 0, y \geq 0, z \geq 0, z \leq 1 - y^2, x + y \leq 1\right\}$

Change of variables

Let φ be a $\mathcal{C}^1$-diffemorphism from Δ to D such that $(u,v,w) \to \varphi(u,v,w) = (x,y,z)$. Then we have

$$\int\int\int_D f(x,y,z)\, dxdydz = \int\int\int_\Delta f \circ \varphi(u,v,w)\, |J_\varphi(u,v,w)|dudvdw$$

where $|J_\varphi|$ is the absolate value of Jacobian.



Integration in cylindrical coordinates

In dimension 3, the cylindrical coordinates are as follow

$$\begin{cases} x = rcos\theta \\ y = rsin\theta \\ z = z \end{cases}$$

The determinant of Jacobian matrix $\varphi : (r, \theta, z) \to (x, y, z)$ is

$$|J_\varphi| = \begin{vmatrix} cos\theta & -rsin\theta & 0 \\ sin\theta & rcos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Then $\int\int\int_D f(x, y, z)\, dxdydz = \int\int\int_\Delta g(r, \theta, z)\, rdrd\theta dz.$

Example 26 *Calculate* $I = \int\int\int_D (x^2 + y^2 + 1)\, dxdydz$
where $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq 2\}$

Example 27 *Evaluate* $I = \int\int\int_D z^{x^2+y^2} dxdydz$
where $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq 1\}$

Integration in spherical coordinates

In dimension 3, the spherical coordinates are defined as follow

$$\begin{cases} x = r sin\theta cos\varphi \\ y = r sin\theta sin\varphi \\ z = r cos\theta \end{cases}$$

The determinant of the jacobian matrix $\varphi : (r,\theta,\varphi) \to (x,y,z)$ is

$$|J_\varphi| = \begin{vmatrix} sin\theta cos\varphi & rcos\theta cos\varphi & -rsin\theta sin\varphi \\ sin\theta sin\varphi & rcos\theta sin\varphi & rsin\theta cos\varphi \\ cos\theta & -rsin\theta & 0 \end{vmatrix} = r^2 sin\theta$$

so $\int\int\int_D f(x,y,z)\, dxdydz = \int\int\int_\Delta g(r,\theta,\varphi)\, r^2 sin\theta dr d\theta d\varphi.$

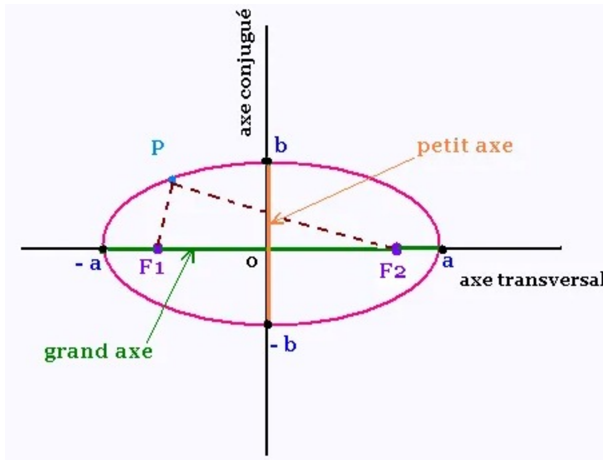
Example 28 *Evaluate the triple integral by changing to spherical coordinates* $I = \int\int\int_D z dxdydz$, *where* $D = \{(x,y,z) \in \mathbb{R}^3, x^2 + y^2 + z^2 \leq R^2, z \geq 0\}$

1.3 Applications of integration: Area and volume

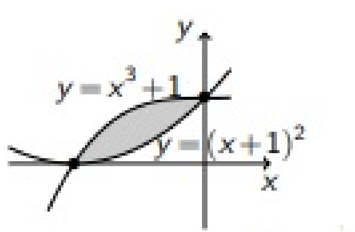
a)**Area of a domain** D

Let D be a subset of $\mathbb{R}^2$, The integral $\int\int_D f(x,y)\,dxdy$ where $f(x,y)=1$ is the area of the domain D.

Example 29 *Calculate the area bounded by the ellipse defined by the equation* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

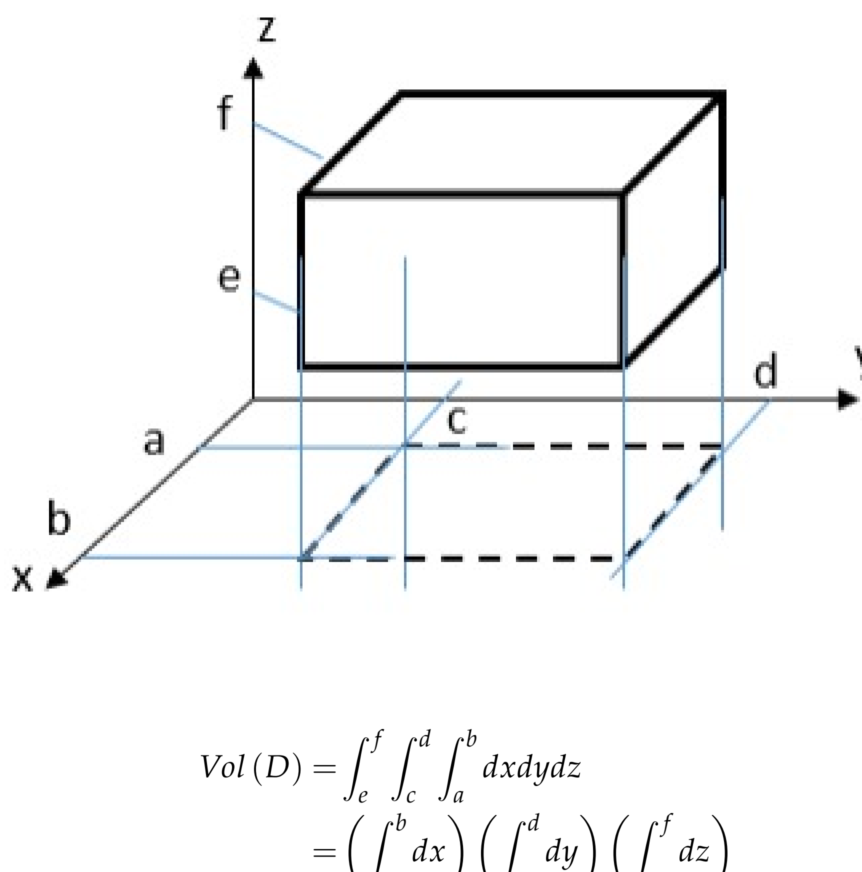


Example 30 *Find the area of the region* $D\subset\mathbb{R}^2$ *bounded by the curves* $y=x^2+2x+1$ *and* $y=x^3+1$.



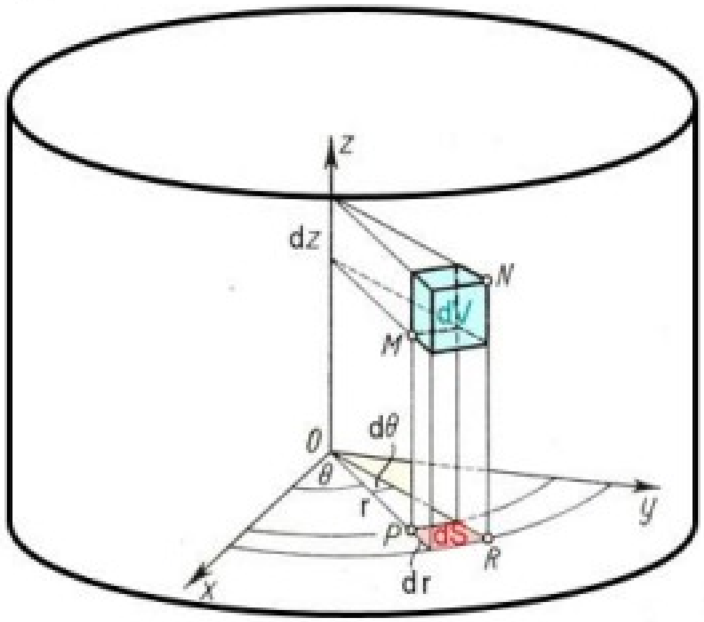
$b)$**Volumes**
The volume of solid is given by $\int\int\int_D dxdydz$ such that D is the domain bounded by the solid.

Example 31 *Find the volume of a cube*



$$Vol(D) = \int_e^f \int_c^d \int_a^b dxdydz$$
$$= \left(\int_a^b dx\right)\left(\int_c^d dy\right)\left(\int_e^f dz\right)$$
$$= (b-a)(c-d)(e-f).$$

Example 32 *Calculate the volume of a cylinder*



By using the cylindrical coordinates, we get

$$\begin{aligned}
Vol\,(D) =& \int_0^h \int_0^{2\pi} \int_0^R r dr d\theta dz \\
=& \left(\int_0^R r dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^h dz\right) \\
=& \left[\frac{r^2}{2}\right]_0^R [\theta]_0^{2\pi} [z]_0^h \\
=& \left(\frac{R^2}{2}\right)(2\pi)(h) \\
=& h\pi R^2.
\end{aligned}$$