Introduction

1 Elements of Mathematical Language

1.1 Elements of Mathematical Language

We can view mathematical language as a construction game, whose goal is to produce true statements. The basic rule of this game is that a mathematical statement can only be true or false. It cannot be "almost true" or "half false." One constraint will therefore be to avoid all ambiguity and each word must have a precise mathematical meaning. Depending on the case, a mathematical statement may have different names:

Assertion: This is the term we use most often to designate a statement about which we can say whether it is true or false.

Expression: This is a set of signs (letters, numbers, symbols, words,...) possessing a meaning in a given context.

Axiom: This is a statement assumed to be true and which we do not seek to prove.

Theorem: This is a statement whose truth must be established.

Corollary: This is a direct consequence of a theorem.

Lemma: This is a preparatory theorem for establishing a theorem of greater importance.

Proposition: This is a term for a proven result less important than a theorem.

Conjecture: This is a statement that we suppose to be true without being able to prove it. It is a plausible hypothesis in view of some examples.

Examples:

Axioms: The axiomatic definition of natural numbers by Peano is usually informally described by five axioms:

- The element called zero and denoted 0 is a natural number.
- Every natural number n has a unique successor denoted $S(n)$ or S_n.
- No natural number has 0 as its successor.
- Two natural numbers with the same successor are equal.
- If a set of natural numbers contains 0 and contains the successor of each of its elements, then this set is equal to N.

Conjecture: (Goldbach's Conjecture) Every even integer strictly greater than 2 is the sum of two prime numbers.

Theorem: Everyone knows the theorems of Thales, Pythagoras and the intermediate value theorem in Analysis.

We have Fermat's Last Theorem: "There do not exist non-zero integers x, y, z when the power n is strictly greater than 2 satisfying the equation: $x^n + y^n = z^n$"

which remained in the state of a conjecture for 350 years before being finally completely proven by Andrew Wiles in 1994.

Expression:

- Let x be a real number, we consider the expression $3x^2 + 4x - 5$.
- In the plane, we consider ABC a triangle.
- Let f be a function defined on R by $f(x) = e^x$.

1.2 Writing Mathematical Proofs

1.2.1 Basic Principles

Mathematical writing aims to clearly make the reader understand a mathematical problem. However, writing, unlike mathematics, is not an exact science, meaning that several writings are possible for the same problem.

In general, the writing of a question should include three parts:

- The introduction
- The reasoning
- The conclusion

Here are some indications to improve writing and learn some useful conventions that are good to know:

1/ Introduce what you are talking about

Introduce all the variables used, even if they are defined in the statement. For example:

- Let $n \in N$.
- For all $n \in N$

We can introduce a personal variable, for example, in the study of a function when the zeros of the derivative have a somewhat long expression and we must draw the variation table.

Example: Let $x_1 = \frac{1+\sqrt{2}}{2}$ and $x_2 = \frac{1-\sqrt{2}}{2}$.

Highlight logical articulations:

Some small words very useful in writing:

- therefore, then, it follows, hence, consequently, thus,
- but, we know that, moreover, furthermore, next, finally,
- but, however, nevertheless, since, as, because,...

These small words allow us to put coherence in our reasoning and make the reading clearer.

Example: Show that $\forall x \in [0,1], \sqrt{1-x^2} \in [0,1]$.

Let $x \in [0,1]$, by the increasing nature of the square function on R_+, we have

$$0 \leq x^2 \leq 1.$$

Consequently

$$0 \leq 1 - x^2 \leq 1.$$

By the increasing nature of the square root function on R_+:

$$0 \leq \sqrt{1-x^2} \leq 1.$$

Consequently: $\forall x \in [0,1], \sqrt{1-x^2} \in [0,1]$.

Announce what you are doing:

Writing a question correctly in mathematics also means explaining what you are doing. Announce the resolution method at the beginning: "Let us show that..., Let us show by contradiction that..., It remains only to show that...".

Cite a definition or a theorem:

Citing a definition or a theorem must be done precisely. You must give clearly and without error the hypotheses, notations and conclusion.

A poorly written theorem, imprecise, a missing hypothesis, all this gives an impression of lack of rigor and can lead to an erroneous conclusion.

Example: Define the derivative number of a function at a point.

Incorrect answer: The derivative number of f at a is:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Lack of precision, who are f and a? Why does the limit of the difference quotient exist?

Correct answer: Let f be a function defined on the interval I and $a \in I$. We say that f is differentiable at a if and only if the limit $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists and is finite.

We call this limit the "derivative number" of f at a which we denote $f'(a)$.

No mixing of genres:

Write in English or in mathematics but not both at once, for example: We write "the sum of two integers is an integer" or "$\forall m, n \in Z, m + n \in Z$" but not: "$\forall m, n \in Z$, the sum of m and n is an integer."

Make the difference between f and $f(x)$:

Incorrect writing: The function $\frac{x}{x^2+1}$ is differentiable on R.

Correct writing: The function $x \mapsto \frac{x}{x^2+1}$ is differentiable on R.

Indeed $\frac{x}{x^2+1}$ is not a function but an algebraic expression; a function is a relation that associates with a quantity x called a variable the quantity $f(x)$. We then denote it $x \mapsto f(x)$.

Showing an implication:

When we want to show that $p \Rightarrow q$, we proceed by one of the following two methods:

1/ We assume that p is true and we show that then q is true.

Example: If n is an odd natural number then the integer $3n+7$ is even.

Let n be an odd natural number, then there exists $k \in N$ such that: $n = 2k+1$. Consequently, we have $3n+7 = 3(2k+1)+7 = 6k+10 = 2(3k+5)$. Therefore there exists $k' \in N$ such that $3n+7 = 2k'$, which proves that the integer $3n+7$ is even.

2/ By contrapositive. We assume that "not q" is true and we show that then "not p" is true.

Example: The classic example of using proof by contrapositive concerns the injectivity of a mapping.

Thus to show that a function $f : E \to F$ is injective, we can show the logical implication: $\forall x_1, x_2 \in E, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

But often it is simpler to show the contrapositive: $\forall x_1, x_2 \in E, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Showing that an implication is false:

To show the implication $p \Rightarrow q$ is false, it suffices to find a counterexample where proposition p is true and proposition q is false.

Example: Consider the proposition "the sequence (U_n) is increasing therefore the sequence (U_n) is divergent".

We must therefore find a counterexample of a sequence that is increasing and convergent. We can easily verify that the sequence (U_n) defined by $U_n = 1 - \frac{1}{n}$ is increasing and its limit is finite.

Showing an equivalence: To show that $p \Leftrightarrow q$, we can proceed in two ways:

Either we reason by equivalence, as is the case in solving equations. Or we reason by double implication: we assume that p is true and we then show that q is true and conversely, we assume that q is true and we show that p is true.

Proof by contradiction:

When we want to show that a property p is true, we can reason by contradiction, i.e. assume p is false and arrive at a contradiction.

Example: Show the irrationality of $\sqrt{2}$.

Writing a proof by induction:

Reasoning by induction obeys the following principle: let P_n be a proposition that depends on a natural number n:

If P_0 is true and if $\forall n \in N : P_n \Rightarrow P_{n+1}$, then $\forall n \in N, P_n$ is true.

Example: Let the sequence (v_n), $v_0 = 10$ and for all $n \in N$: $v_{n+1} = \sqrt{v_n + 6}$. Prove that the proposition P_n: $3 \leq v_n \leq 10$ is true for all $n \in N$.

1.3 Expression "Without Loss of Generality"

It happens that two or more cases in a proof are similar and writing them separately seems repetitive or unnecessary.

Example: Show that if two integers have different parities then their sum is an odd integer.

Proof: Let m and n be two integers of different parities, we must show that $m+n$ is an odd integer.

We have two cases:

- Case 1: Suppose that m is even and n is odd, then there exist two integers a and b such that $m = 2a$ and $n = 2b+1$. Hence we obtain $m+n = 2(a+b)+1$ which is odd.

- Case 2: Suppose now that m is odd and n is even, then there exist two integers a and b such that $m = 2a+1$ and $n = 2b$. Therefore we obtain $m+n = 2(a+b)+1$ which is odd.

We note that the two cases are treated in the same way. In general in mathematics, we avoid this repetition by using the expression "without loss of generality", the previous proof would be for example:

Let m and n be two integers of different parities, we must show that $m+n$ is an odd integer.

Without loss of generality, suppose that m is even and n is odd, then there exist two integers a and b such that $m = 2a$ and $n = 2b+1$. Hence we obtain $m+n = 2(a+b)+1$ which is odd.

1.4 Constructive Proofs and Existential Proofs

Existential Proofs

Proposition 1.1 *There exist two irrational numbers x and y such that x^y is rational.*

We know that $\sqrt{2}$ is irrational. We then consider the number $\sqrt{2}^{\sqrt{2}}$ which is either rational or irrational.

If $\sqrt{2}^{\sqrt{2}}$ is rational, the proposition is proven by considering $x = y = \sqrt{2}$.

If $\sqrt{2}^{\sqrt{2}}$ is irrational, then by setting $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$, we then obtain $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$ and the proposition is proven.

The proof of the existence of two irrational numbers x and y such that x^y is rational is done without being able to give an example of two irrational numbers that satisfy $x^y \in Q$.

This type of proof is called a "non-constructive proof" or "existential proof" in mathematical language.

Constructive Proofs

Proposition 1.2 *There exist two irrational numbers x and y such that x^y is rational.*

Let $x = \sqrt{3}$ and $y = \log_3(4)$. x and y are irrational and we have:

$$x^y = \sqrt{3}^{\log_3(4)} = 3^{\frac{1}{2}\log_3(4)} = 3^{\log_3(4)^{\frac{1}{2}}} = 3^{\log_3(2)} = 2$$