Chapter 2 : System of linear algebraic equations

1 Definition

A simple system of linear algebraic equations in the variables x and y it can be expressed as follows : $ax+by=c$, where $a,\ b$ and c are real numbers. The graph of this linear equation with two variables is a stright line $y=-\frac{a}{b}x+\frac{c}{b}$. A system of two linear equations with two unknowns or variables $x_1$and x_2 is given by

$$\begin{aligned} a_1x_1+b_2x_2 &= c_1 \\ a_2x_1+b_2x_2 &= c_2 \end{aligned}$$

This system it can be rewritten as in the form $AX=c$ as follows :

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

In general, the system of m linear equations and with n unknowns hase the following form :

$$\begin{gathered} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=c_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=c_2 \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=c_m \end{gathered}$$

Also, this system is in the form $AX=c$ where A is a $m\times n$ matrix and c is $m\times 1$ vector such that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

- In the case where $c_1=c_2=...=c_n=0$, the system is said to be homogeneous, otherwise it is nonhomogeneous.

Example :

Consider this system of linear equations :

$$\begin{aligned} 2x_1+3x_2+4x_3 &= 0 \\ 4x_1-2x_2 \qquad &= 0 \end{aligned}$$

This system is rewriten as

$$\begin{pmatrix} 2 & 3 & 4 \\ 4 & -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

2 Solution of a linear system

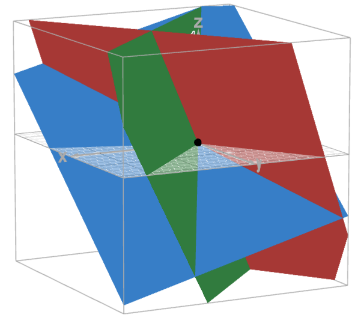
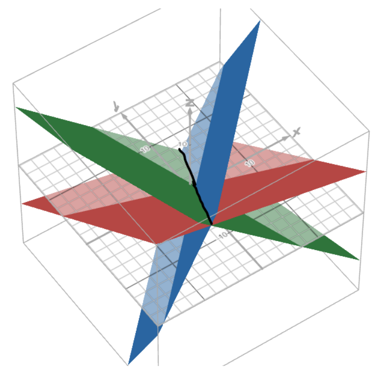
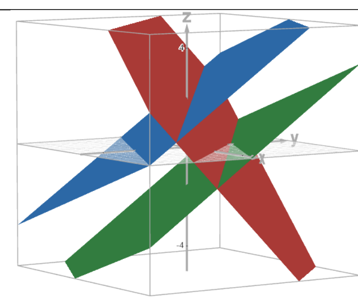
The set $x_1, x_2, ..., x_n$is a solution of linear system if it satisfies each equation in the system. For example the following system

$$\begin{gathered} 2x_1 + x_2 = 0.....(1) \\ 3x_1 - 2x_2 = 1.......(2) \end{gathered}$$

it can be solved as follows : From (1) we get $x_2 = -2x_1$ by replacing in (2) we get $x_1 = \frac{1}{7}$ and then $x_2 = -\frac{2}{7}$. The two equations in the system represent an equation of a stright line and the solution $(x_1 = \frac{1}{7}, x_2 = -\frac{2}{7})$ is the coordinates of the intersection point of the two stright lines.

- If a system of 2 linear equations in 2 unknowns has infinitely many solutions, then the two straight lines represented by the equations are identical. If the system has no solution, then the two straight lines are parallel but distinct.

Consistent and inconsistent system of linear equations :

System	Example	Graph
Consistent system : is a system with one solution or Infinitely many solutions	One solution : $x_1 + 3x_2 + 4x_3 = 0$ $2x_1 - 2x_2 - x_3 = 0$ $3x_1 + 2x_2 - x_3 = 0$	
	Infinite solutions : $x_1 + 3x_2 + 4x_3 = 1$ $2x_1 - 2x_2 - x_3 = 2$ $3x_1 + x_2 + 3x_3 = 3$	
Inconsistent system : is a system with no solutions	$x_1 + x_2 + x_3 = 2$ $x_1 + 2x_2 - 3x_3 = -5$ $2x_1 + 4x_2 - 6x_3 = 9$	

Trivial solution :

The homogeneous system of linear equations :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m \end{aligned}$$

always has a trivial solution $x_1 = x_2 = \cdots = x_n = 0$.

3 Methods for solving Linear system

3.1 Gaussian elimination

Consider the following linear system :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m \end{aligned}$$

This system it can be solved using gauss methode by following these steps :

1. Writing the system in augmented matrix form $(A|c)$as follows :

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & c_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & c_m \end{array}\right)$$

2. Applying the following elementary rows operations on the augmented matrix :
   - Swap two rows $R_i \longleftrightarrow R_j$
   - Multiply a row by a nonzero scalar $R_i \longrightarrow kR_i$
   - Add or subtract a multiple of one row to from another row $R_i \longrightarrow R_i + kR_j$
3. Repeat these operations until we arrive at row-equivalent augmented matrix
4. Back-substitute to find the values of the unknowns

Example 1.

Consider the following linear system :

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 1 \\ 5x_1 + x_2 - 2x_3 &= 1 \end{aligned}$$

This system is rewritten in augmented form as

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & -1 & 1 \\ 5 & 1 & -2 & 1 \end{array}\right)$$

Next, we perform elementary row operations on this system

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & 1 & -1 & 1 \\ 5 & 1 & -2 & 1 \end{array}\right) \overset{R_2 \longrightarrow R_2 - 2R_1}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -5 \\ 5 & 1 & -2 & 1 \end{array}\right)$$

$$\overset{R_3 \longrightarrow R_3 - 5R_1}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -5 \\ 0 & -4 & -7 & -14 \end{array}\right) \overset{R_3 \longrightarrow R_3 - 4R_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -5 \\ 0 & 0 & 5 & 6 \end{array}\right)$$

$$\overset{R_3 \longrightarrow \frac{1}{5} R_3}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -3 & -5 \\ 0 & 0 & 1 & \frac{6}{5} \end{array} \right)$$

then we get

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ -x_2 - 3x_3 &= -5 \\ x_3 &= \frac{6}{5} \end{aligned}$$

Back-substitute gives $-x_2 - 3\left(\frac{6}{5}\right) = -5 \Longrightarrow x_2 = \frac{19}{5}$

$x_1 + \frac{19}{5} + \frac{6}{5} = 3 \Longrightarrow x_1 = 2$

Example 2.

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_1 + 3x_2 - x_3 &= 2 \\ 2x_1 + 6x_2 - 2x_3 &= 1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 3 & -1 & 2 \\ 2 & 6 & -2 & 1 \end{array} \right) \overset{\substack{R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - 2R_1}}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 4 & 0 & 1 \end{array} \right)$$

$$\overset{R_3 \longrightarrow R_3 - 2R_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & -3 \end{array} \right)$$

The third equation (R_3) in the last form of augmented matrix leads to a contraduction $0 = -3$ which means that the system does not has a solution (inconsistent).

Example 3.

$$\begin{aligned} x_1 + x_2 - 2x_3 &= 2 \\ 2x_1 - x_2 + 3x_3 &= 1 \\ 3x_1 \quad + \quad x_3 &= 3 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 2 & -1 & 3 & 1 \\ 3 & 0 & 1 & 3 \end{array} \right) \overset{\substack{R_2 \longrightarrow R_2 - 2R_1 \\ R_3 \longrightarrow R_3 - 3R_1}}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & -3 & 7 & -3 \\ 0 & -3 & 7 & -3 \end{array} \right)$$

$$\overset{R_3 \longrightarrow R_3 - R_2}{\longrightarrow} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & -3 & 7 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right) \Longrightarrow \begin{aligned} x_1 + x_2 - 2x_3 &= 2 ....(1) \\ -3x_2 + 7x_3 &= -3 ......(2) \\ 0x_1 + 0x_2 + 0x_3 &= 0 ....(3) \end{aligned}$$

The third equation (3) in the last form of augmented matrix leads to $0x_1+0x_2+0x_3 = 0$. As is clear, any value of $x_1,\ x_2$ and x_3 is a solution to this equation, which indicates that there are infinitely many solutions to this system. We set $x_3 = t$ and by replacing in (1) and (2) we get $x_2 = 1 + \frac{7}{3}t,\ x_1 = 1 - \frac{t}{3}$.

3.2 Inverse matrix method

As we learn from chapter 2 that the inverse matrix is given for square matrices as well as $AA^{-1} = A^{-1}A = I$. Thus the methode of inverse matrix is used to solve a system where the number of equations is equal the number of unknowns as follows :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= c_n \end{aligned}$$

This can be written as :

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

which has the form $AX = c$ by multiplying this by $A^{-1}$from the left we get $A^{-1}AX = A^{-1}c$, i.e.,$X = A^{-1}c$ written as

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}^{-1} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

This means that if we need to solve a system of linear equation has the form $AX = c$ where A is a square matrix, we need to find A^{-1} the inverse matrix of A.

Example

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ -2x_1 - 7x_2 - 2x_3 &= 11 \\ -3x_1 - x_2 + 2x_3 &= 3 \end{aligned}$$

The system in the form $AX = c$:

$$\begin{pmatrix} 1 & 2 & -1 \\ -2 & -7 & -2 \\ -3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \\ 3 \end{pmatrix}$$

which gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ -2 & -7 & -2 \\ -3 & -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 11 \\ 3 \end{pmatrix}$$

we have

$$A^{-1} = \begin{pmatrix} 1 & 2 & -1 \\ -2 & -7 & -2 \\ -3 & -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -16 & -3 & -11 \\ 10 & -1 & 4 \\ -19 & -5 & -3 \end{pmatrix}$$

By replacing in the previous equation we get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -16 & -3 & -11 \\ 10 & -1 & 4 \\ -19 & -5 & -3 \end{pmatrix} \begin{pmatrix} 2 \\ 11 \\ 3 \end{pmatrix}$$

and then we find

$$x_1 = \frac{1}{23}(-32 - 33 - 33) = -\frac{98}{23}$$
$$x_2 = \frac{1}{23}(20 - 11 + 12) = \frac{21}{23}$$
$$x_3 = \frac{1}{23}(-38 - 55 - 9) = -\frac{102}{23}$$

Remark on the Method

As known the inverse of a matrix A is defined only if $det(A) \neq 0$, then the solution of linear system with n equations and n unknowns is depend on the determinant :

- If $det(A) \neq 0$, then the system has a unique solution
- If $det(A) = 0$, then the system can have no solution, or infinitely many solutions.

3.3 Cramer's Rule

Cramer's method for solving a system of linear equations is based on calculating determinants which is defined for square matrices. Thus, the method is used to solve a system with n equations and n unknowns. For a system of linear equations $AX = c$ expressed as :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= c_n \end{aligned}$$

The solution is given by

$$x_k = \frac{det(A_k)}{det(A)}$$

Where $k = 1, 2, \cdots n$ and the matrix A_k is obtained from the original matrix A by replacing its $k - th$ column with c vector. For example

$$A_1 = \begin{pmatrix} c_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ c_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}, A_2 = \begin{pmatrix} a_{11} & c_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & c_2 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & c_n & a_{n3} & \cdots & a_{nn} \end{pmatrix} \ldots$$

Example :

$$\begin{aligned} -x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + x_2 - 2x_3 &= 4 \\ x_1 + 3x_2 + x_3 &= 2 \end{aligned}$$

This sysem is writen in the firm $AX = c$ as $\begin{pmatrix} -1 & 2 & 1 \\ 3 & 1 & -2 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, where $det(A) = -9 \neq 0$, this implies the system hase a unique solution given by :

$$x_1 = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 4 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}}{|A|} = -\frac{1}{9}$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} -1 & 1 & 1 \\ 3 & 4 & -2 \\ 1 & 2 & 1 \end{vmatrix}}{|A|} = \frac{11}{9}$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} -1 & 2 & 1 \\ 3 & 1 & 4 \\ 1 & 3 & 2 \end{vmatrix}}{|A|} = -\frac{14}{9}$$