Abdelhafid Bousouf University Center, Mila
Institute of Mathematics and Computer Science
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Module : *Algebra 3*
Course leader : *Dr. Haddad.N*

WORKSHEET 2 (DIAGONALIZATION OF ENDOMORPHISMS)

Exercise 1

Let f be an endomorphism of $\mathbb{R}^3$ defined by

$$\begin{array}{rcl} f: & \mathbb{C}^3 & \to \mathbb{C}^3 \\ & (x,y,z) & \mapsto (2x+2y,-x,z) \end{array}$$

1. Find the eigenvalues of f and the eigenspaces associated to these eigenvalues.
2. Give a basis of each eigenspace of f.

Exercise 2

Let g be an endomorphism of $\mathbb{R}_2[X]$ defined by

$$\begin{array}{rcl} g: & \mathbb{R}_2[X] & \to \mathbb{R}_2[X] \\ & P & \mapsto (1-X)P' \end{array}$$

1. Find the eigenvalues of g and the eigenspaces associated to these eigenvalues.
2. Give a basis of each eigenspace of g.

Exercise 3

Determine the characteristic polynomials and eigenvalues of the following matrices :

$$A_1 = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ -2 & 0 & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -4 & 0 & -2 \\ 0 & 1 & 0 \\ 5 & 1 & 3 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 3 & -4 & 0 & 2 \\ 4 & -5 & -2 & 4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{pmatrix}.$$

Exercise 4

Let f be an endomorphism of $\mathbb{R}^n$ such that $f \circ f = f$. Suppose that f is different from both the zero map and the identity.

1. Compute the eigenvalues of f.

2. Show that for all $x \in \mathbb{R}^n$, the vectors $u = f(x)$ and $v = x - f(x)$ are eigenvectors of f. Specify the eigenvalues associated to u and v.
3. Deduce that f is diagonalizable.
4. Express the eigenspaces in terms of $\ker(f)$ and $\mathrm{Im}(f)$.

Exercise 5

Let A_α be the following matrix

$$A_\alpha = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 2-\alpha & \alpha-2 & \alpha \end{pmatrix}.$$

1. Determine and factorize the characteristic polynomial of A_α.
2. Determine the values of α for which A_α is diagonalizable.
3. Suppose $\alpha = 2$, determine a diagonal matrix D and an invertible matrix P such that $A_2 = PDP^{-1}$.

Exercise 6

1. Let f be an endomorphism of a finite-dimensional K-vector space E. Show that 0 is in the spectrum of f if and only if f is not surjective.
2. Show that if $A \in M_n(K)$ is diagonalizable, then so is A^k for any integer k.

Exercise 7 (*Supplementary*)

Let $A \in M_3(\mathbb{R})$ be the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -2 & 0 & -2 \\ 3 & 3 & 5 \end{pmatrix}.$$

1. Calculate the characteristic polynomial of A.
2. Determine the eigenvalues of A and show that A is diagonalizable.
3. Find an invertible matrix P such that $D = P^{-1}AP$ is diagonal.

Exercise 8 (*Supplementary*)

Let f be the application

$$f : \begin{array}{ccc} \mathbb{C}^3 & \to & \mathbb{C}^3 \\ (x,y,z) & \mapsto & (2x+2y, -x, z) \end{array}$$

1. Show that f is an endomorphism.
2. Calculate the matrix associated to f with respect to the canonical basis of $\mathbb{R}^3$.
3. Calculate the characteristic polynomial $P_f(\lambda)$.
4. Determine the eigenvalues of f and show that f is diagonalizable.