CHAPTER 1

Diagonalization of Endomorphisms

In this chapter, let E be a vector space over $K = \mathbb{R}$ or $\mathbb{C}$, and let f be an endomorphism of E.

1.1 Eigenvectors and Eigenvalues

Definition 1.1.1 *Let v be a vector in E. We say that v is an eigenvector of f if*

- $v \neq 0$,
- *There exists $\lambda \in K$ such that*

$$f(v) = \lambda v.$$

The scalar λ is called the eigenvalue associated to v.
The pair (λ, v) is called an eigenpair of f.

Definition 1.1.2 *The set of eigenvalues of f, denoted* $\mathrm{Sp}(f)$*, is called the spectrum of f, that is*

$$\mathrm{Sp}(f) = \{\lambda \in K : \exists v \in E \setminus \{0\}, f(v) = \lambda v\}.$$

Proposition 1.1.1 *Let λ be an eigenvalue of f. The set of all eigenvectors associated with λ, together with the zero vector 0_E forms a vector subspace of E, called the eigenspace of f associated with λ, denoted E_λ, i.e.,*

$$E_\lambda = \{v \in E : f(v) = \lambda v\}.$$

Proof. We have $0_E \in E_\lambda$. Let $v_1, v_2 \in E_\lambda$ and $\alpha, \beta \in K$.

$$\begin{aligned} f(\alpha v_1 + \beta v_2) &= \alpha f(v_1) + \beta f(v_2) \\ &= \alpha\lambda v_1 + \beta\lambda v_2 \\ &= \lambda(\alpha v_1 + \beta v_2), \end{aligned}$$

so $\alpha v_1 + \beta v_2 \in E_\lambda$. Thus, E_λ is a vector subspace of E. ■

Example 1.1.1 1. *Let* $\mathrm{Id}_E : E \to E$ *be the identity endomorphism defined by*

$$\mathrm{Id}_E(x) = x.$$

Then λ *is an eigenvalue of* Id_E *if and only if there exists* $v \neq 0$*, such that*

$$\mathrm{Id}_E(v) = \lambda v,$$

which means

$$v = \lambda v,$$

thus $\lambda = 1$ *and* $\mathrm{Sp}(\mathrm{Id}_E) = \{1\}$*. The eigenspace of* Id_E *corresponding to* $\lambda = 1$ *is*

$$E_1 = \{v \in E : \mathrm{Id}_E(v) = v\} = E.$$

2. *Let* $g : \mathbb{R}^2 \to \mathbb{R}^2$ *be the mapping defined by*

$$g(x, y) = (3x, 5y).$$

λ *is an eigenvalue of* g *if and only if there exists* $(x, y) \neq (0, 0)$ *such that*

$$g(x, y) = \lambda(x, y).$$

This implies the system

$$\begin{cases} 3x = \lambda x \\ 5y = \lambda y \end{cases},$$

so if $x \neq 0$ *and* $y = 0$*,* $\lambda = 3$*, if* $y \neq 0$ *and* $x = 0$*,* $\lambda = 5$*. Therefore, the eigenvalues of* g *are 3 and 5, so the spectrum is*

$$\mathrm{Sp}(g) = \{3, 5\}.$$

The eigenspace for $\lambda = 3$ is

$$\begin{aligned} E_3 &= \{(x,y) \in \mathbb{R}^2 : g(x,y) = 3(x,y)\} \\ &= \{(x,y) \in \mathbb{R}^2 : 3x = 3x, 5y = 3y)\} \\ &= \{(x,y) : x \in \mathbb{R}, y = 0\} \\ &= \{(x,0) : x \in \mathbb{R}\} \\ &= \langle (1,0) \rangle, \end{aligned}$$

and for $\lambda = 5$

$$\begin{aligned} E_5 &= \{(x,y) \in \mathbb{R}^2 : g(x,y) = 5(x,y)\} \\ &= \{(x,y) \in \mathbb{R}^2 : 3x = 5x, 5y = 5y)\} \\ &= \{(x,y) : x = 0, y \in \mathbb{R}\} \\ &= \{(0,y) : y \in \mathbb{R}\} \\ &= \langle (0,1) \rangle, \end{aligned}$$

Proposition 1.1.2 *The eigenvalue associated with an eigenvector is unique.*

Proof. Suppose λ_1 et λ_2 are two eigenvalues associated to the eigenvector v, with $\lambda_1 \neq \lambda_2$. Then

$$f(v) = \lambda_1 v \quad \text{and} \quad f(v) = \lambda_2 v.$$

By subtracting, we obtain

$$\lambda_1 v - \lambda_2 v = 0$$

so, since v is nonzero, it follows that $\lambda_1 - \lambda_2 = 0$, contradiction with $\lambda_1 \neq \lambda_2$ ■

Remark 1.1.1 *The eigenvector associated with an eigenvalue is not unique. Indeed, if v is an eigenvector associated with the eigenvalue λ, then for any scalar k, kv is also an eigenvector associated with λ since*

$$f(kv) = kf(v) = k\lambda v = \lambda(kv).$$

Theorem 1.1.1 *The dimension of the eigenspace E_λ associated with the eigenvalue λ is less than or equal to the multiplicity m of this eigenvalue, i.e.,*

$$\dim(E_\lambda) \leq m.$$

Proof. Let $l = \dim(E_\lambda)$. Then, there exist l vectors $v_1, v_2, \ldots, v_l$ forms a basis of E_λ. These vectors satisfy by definition

$$f(v_i) = \lambda v_i, \quad i = 1, 2, \ldots, l.$$

Complete this linearly independent system $\{v_1, \ldots, v_l\}$ into a basis of the vector space E by adding vectors $v_{l+1}, \ldots, v_n$. The matrix representation B of f in this basis has the form

$$B = \begin{pmatrix} \lambda & 0 \ldots 0 & b_{1,l+1} \ldots b_{1,n} \\ 0 & \lambda \ldots 0 & b_{2,l+1} \ldots b_{2,n} \\ \vdots & & \\ 0 & 0 \ldots \lambda & b_{l,l+1} \ldots b_{l,n} \\ 0 & 0 \ldots 0 & b_{l+1,l+1} \ldots b_{l+1,n} \\ \vdots & & \\ 0 & 0 \ldots 0 & b_{n,l+1} \ldots b_{n,n} \end{pmatrix}.$$

The characteristic polynomial of f, computed in this new basis, is the following

$$\begin{aligned} P(X) &= \begin{vmatrix} \lambda - X & 0 \ldots 0 & b_{1,l+1} \ldots b_{1,n} \\ 0 & \lambda - X \ldots 0 & b_{2,l+1} \ldots b_{2,n} \\ \vdots & & \\ 0 & 0 \ldots \lambda - X & b_{l,l+1} \ldots b_{l,n} \\ 0 & 0 \ldots 0 & b_{l+1,l+1} - X \ldots b_{l+1,n} \\ \vdots & & \\ 0 & 0 \ldots 0 & b_{n,l+1} \ldots b_{n,n} - X \end{vmatrix} \\ &= (\lambda - X)^l \begin{vmatrix} b_{l+1,l+1} - X \ldots b_{l+1,n} \\ \vdots \\ b_{n,l+1} \ldots b_{n,n} - X \end{vmatrix} \end{aligned}$$

hence

$$l \leq m.$$

■

1.2 Characterization of Eigenvalues

Proposition 1.2.1 *A scalar $\lambda \in K$ is an eigenvalue of f if and only if $f - \lambda \mathrm{Id}_E$ is not injective.*

Proof. Let $\lambda \in K$

$$\begin{aligned}
\lambda \in \mathrm{Sp}(f) &\Leftrightarrow \exists v \in E,\, v \neq 0_E : f(v) = \lambda v, \\
&\Leftrightarrow \exists v \in E,\, v \neq 0_E : f(v) - \lambda v = 0_E, \\
&\Leftrightarrow \exists v \in E,\, v \neq 0_E : f(v) - \lambda \mathrm{Id}_E(v) = 0_E. \\
&\Leftrightarrow \exists v \in E,\, v \neq 0_E : v \in \ker(f - \lambda \mathrm{Id}_E) \\
&\Leftrightarrow \ker(f - \lambda \mathrm{Id}_E) \neq \{0_E\} \\
&\Leftrightarrow f - \lambda \mathrm{Id}_E \; is\ not\ injective.
\end{aligned}$$

■

1.3 Finite-Dimensional Case

Let E be a vector space of dimension n over $K = \mathbb{R}$ or $\mathbb{C}$. We consider a basis $\mathcal{B}$ of E, and let $A = \mathrm{Mat}_{\mathcal{B}}(f) \in M_n(K)$ be the associated matrix of f with respect to the basis $\mathcal{B}$.

1.3.1 Eigenvectors and eigenvalues of a square matrix

Definition 1.3.1 *Let $A \in \mathcal{M}_n(K)$, A vector $X \in \mathcal{M}_{n,1}(K)$ is called an eigenvector of A if*

- $X \neq 0$,
- *There exists $\lambda \in K$ such that*

$$AX = \lambda X.$$

The scalar λ is called the eigenvalue associated to X.
The pair (λ, X) is called an eigenpair of f.

Definition 1.3.2 *The set of all eigenvalues of A, denoted* $\mathrm{Sp}(A)$, *is called the spectrum of A, i.e.,*

$$\mathrm{Sp}(A) = \{\lambda \in K : \exists X \in \mathcal{M}_{n,1}(K),\, AX = \lambda X\}.$$

Theorem 1.3.1 *Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of A. If X_i is an eigenvector corresponding to λ_i, for all i, then the vectors X_1, X_2, $\ldots$, X_k are linearly independent.*

Proof. (By induction on k):

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of A and let X_1, X_2, ..., X_k, be the corresponding eigenvectors respectively. We shall prove by induction on k that

$$\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_k X_k = 0 \implies \alpha_1 = \alpha_2 = \ldots = \alpha_k = 0,$$

with $\alpha_1, \alpha_2, ..., \alpha_k \in K$.

*) For $k = 1$, the statement holds since the eigenvector X_1 is nonzero.

*) Assume that the statement holds for $k-1$, let $\alpha_1, \alpha_2, ..., \alpha_k \in K$ such that

$$\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_k X_k = 0.$$

By multiplying on the left by the matrix A and use the eigenvector property, we obtain

$$\alpha_1 \lambda_1 X_1 + \alpha_2 \lambda_2 X_2 + \cdots + \alpha_k \lambda_k X_k = 0.$$

Subtract the first relation multiplied by λ_k, we obtain

$$\alpha_1(\lambda_1 - \lambda_k)X_1 + \alpha_2(\lambda_2 - \lambda_k)X_2 + \cdots + \alpha_{k-1}(\lambda_{k-1} - \lambda_k)X_{k-1} = 0.$$

From the hypothesis, the vectors $X_1, X_2, ..., X_{k-1}$, are linearly independent, since the coefficients $(\lambda_i - \lambda_k)$ are nonzero (because the eigenvalues λ_i are distinct), it follows that

$$\alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0.$$

Substituting these values into the original equation, we obtain $\alpha_k X_k = 0$, thus $\alpha_k = 0$. Hence the vectors $X_1, X_2, ..., X_k$, are linearly independent. ■

1.4 Characteristic Polynomial

Definition 1.4.1 *For $A \in \mathcal{M}_n(K)$, the characteristic polynomial P_A is defined as*

$$P_A(\lambda) := \det(A - \lambda I_n).$$

The characteristic equation of A is

$$P_A(\lambda) = 0.$$

Definition 1.4.2 *For an endomorphism f of E, the characteristic polynomial P_f is the characteristic polynomial of any matrix associated to f relative to any basis of E.*

Example 1.4.1 *1. Let $A = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$. The characteristic polynomial of the matrix A is*

$$P_A(\lambda) = det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 1.$$

2. *Let the endomorphism*

$$\begin{aligned} f: \quad & \mathbb{R}^3 && \to \mathbb{R}^3 \\ & (x,y,z) && \mapsto (x-y, 2x+z, x+y+2z) \end{aligned}$$

Let $B = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ *be the canonical basis of* $\mathbb{R}^3$. *The matrix of* f *in* B *is*

$$M = \mathrm{Mat}_B(f) = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then, the characteristic polynomial of f *is*

$$P_f(\lambda) = P_M(\lambda) = \det(M - \lambda I_3) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ 2 & -\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = \lambda^3 - 3\lambda^2 + 3\lambda - 2.$$

Theorem 1.4.1 *Let* $A \in \mathcal{M}_n(K)$, $\lambda \in K$. *Then* λ *is an eigenvalue of* A *if and only if* λ *is a root of the characteristic polynomial* P_A, *i.e.,*

$$\lambda \in Sp(A) \iff P_A(\lambda) = 0.$$

Proof. Let $\lambda \in K$

$$\begin{aligned} \lambda \in \mathrm{Sp}(A) \quad & \Leftrightarrow \quad \lambda \in \mathrm{Sp}(f), \\ & \Leftrightarrow \quad f - \lambda \mathrm{Id}_E \ \textit{is not injective}, \\ & \Leftrightarrow \quad f - \lambda \mathrm{Id}_E \ \textit{is not bijective}, \\ & \Leftrightarrow \quad f - \lambda \mathrm{Id}_E \ \textit{is not invertible}, \\ & \Leftrightarrow \quad A - \lambda \mathrm{Id}_E \ \textit{is not invertible}, \\ & \Leftrightarrow \quad P_A(\lambda) = \det(A - \lambda I_n) = 0 \end{aligned}$$

■

Definition 1.4.3 *The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic equation (*$mult_a(\lambda)$*).*

Definition 1.4.4 *The geometric multiplicity of an eigenvalue is the dimension of the corresponding eigenspace* E *(*$mult_g(\lambda) = \dim(E_\lambda)$*)*

Example 1.4.2 *1. Let*

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C}).$$

The characteristic polynomial of A *is*

$$P_A(\lambda) = \det(A - \lambda I_2) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

Therefore,

$$\lambda \text{ is an eigenvalue of } A \iff P_A(\lambda) = 0 \iff \lambda = \pm i,$$

then $Sp(A) = \{i, -i\}$ *and* $mult_a(i) = mult_a(-i) = 1$, *hence* $mult_g(i) = mult_g(-i) = 1$.

2. Let

$$B = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \in M_3(\mathbb{R}).$$

The characteristic polynomial of B *is*

$$P_B(\lambda) = \det(B - \lambda I_3) = (1 - \lambda)(2 + \lambda)^2.$$

Hence,

$$\lambda \text{ is an eigenvalue of } B \iff P_B(\lambda) = 0 \iff \lambda = 1 \text{ or } \lambda = -2,$$

then $Sp(B) = \{1, -2\}$, $mult_a(1) = 1$ *and* $mult_a(-2) = 2$, *hence* $mult_g(1) = 1$, $mult_g(-2) \leq 2$.

Remark 1.4.1 *The eigenvalues of a triangular or diagonal matrix are its diagonal elements. Indeed, if* $C \in \mathcal{M}_n(K)$ *is a lower triangular matrix, then*

$$P_C(\lambda) = \det(C - \lambda I_n) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda).$$

Thus,

$$Sp(C) = \{ a_{11}, a_{22}, \ldots, a_{nn} \}.$$

Example 1.4.3 *1. Let*

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}).$$

Then

$$Sp(A) = \{ 3, -1, 2 \}.$$

2. *Let*

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1+i & 0 & 0 \\ 2i & -3 & 0 & 0 \\ -2 & i-1 & 5 & 2 \end{pmatrix} \in M_4(\mathbb{C}).$$

Then

$$Sp(B) = \{2, 1+i, 0\}.$$

1.5 Diagonalization of Endomorphisms

In this section, we shall find a basis of E such that the matrix representing the endomorphism f takes, in this basis, its simplest possible form, namely a diagonal matrix.

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

This implies that there exists a basis $\{u_1, u_2, \ldots, u_n\}$ of E such that

$$f(u_i) = \lambda_i u_i, \quad i = 1, \ldots, n.$$

Definition 1.5.1 *An endomorphism f of E is called diagonalizable if there exists a basis $\mathcal{B}$ of E such that the matrix associated with f relative to $\mathcal{B}$ either diagonal.*

Definition 1.5.2 *Two matrices A and B are said to be similar if there exists an invertible matrix P such that $B = P^{-1}AP$.*

Definition 1.5.3 *A matrix $A \in M_n(K)$ is called diagonalizable if it is similar to a matrix diagonal, in other words, if there exists an invertible matrix $P \in M_n(K)$ such as the matrix $D = P^{-1}AP$ either diagonal.*

1.5.1 Necessary and Sufficient Condition for Diagonalization

Theorem 1.5.1 *An endomorphism f (or matrix A) is diagonalizable if and only if the following two conditions are satisfied:*

1. *The characteristic polynomial is split on K (i.e., can be written as a product of linear factors).*

2. *The algebraic and geometric multiplicity of each eigenvalue are equal.*

1.5.2 Sufficient Condition for Diagonalization

Proposition 1.5.1 *If a matrix A has n distinct eigenvalues in K, then A is diagonalizable.*

Remark 1.5.1 *This proposition is only a sufficient condition, not necessary.*

1.5.3 Practical Procedure for Diagonalization

1. Compute the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I_n)$ and find its roots.
2. If all roots (eigenvalues) lie in K, find the eigenspace for each eigenvalue of A (solve the system $(A - \lambda I_n)X = 0$).
3. Verify that the dimension of each eigenspace equals the multiplicity of the corresponding eigenvalue ($mult_a(\lambda_i) = mult_g(\lambda_i), \forall i$).
4. Form the matrix P whose columns are the eigenvectors of A (the matrix P is formed exactly by n linearly independent eigenvectors of A).
5. Compute P^{-1} and obtain the diagonal matrix $D = P^{-1}AP$.

1.5.4 Examples of Diagonalization

Example 1.5.1 *Let the matrix*

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 3 & -2 & 0 \\ -2 & 2 & 1 \end{pmatrix} \in M_3(\mathbb{R}).$$

1. *Determine the characteristic polynomial of A:*

$$P_A(\lambda) = \det \begin{pmatrix} -\lambda & 2 & -1 \\ 3 & -2-\lambda & 0 \\ -2 & 2 & 1-\lambda \end{pmatrix} = (1-\lambda)(\lambda^2 + 2\lambda - 8) = (1-\lambda)(\lambda+4)(\lambda-2).$$

   The eigenvalues of A are 1, -4, and 2.

2. *Show that A is diagonalizable:*

   Since A has three distinct real eigenvalues, it is diagonalizable.

3. *Determine a basis for each eigenspace, and consequently determine the translation matrix P:*

We have

$$E_\lambda = (x, y, z) \in \mathbb{R}^3 : A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} \}.$$

For $\lambda = 1$*:*

$$\begin{cases} 2y - z = x \\ 3x - 2y = y \\ -2x + 2y + z = z \end{cases} \iff x = y = z,$$

so

$$E_1 = \{(x, x, x) : x \in \mathbb{R}\} = \langle (1, 1, 1) \rangle.$$

For $\lambda = 2$*:*

$$\begin{cases} 2y - z = 2x \\ 3x - 2y = 2y \\ -2x + 2y + z = 2z \end{cases} \iff \begin{cases} z = -\frac{1}{2}x \\ y = \frac{3}{4}x \end{cases}$$

so

$$E_2 = \left\{ \left(x, \frac{3}{4}x, -\frac{1}{2}x \right) : x \in \mathbb{R} \right\} = \langle (4, 3, -2) \rangle.$$

For $\lambda = -4$*:*

$$\begin{cases} 2y - z = -4x \\ 3x - 2y = -4y \\ -2x + 2y + z = -4z \end{cases} \iff \begin{cases} z = x \\ y = -\frac{3}{2}x \end{cases}$$

so

$$E_{-4} = \{(x, -\frac{3}{2}x, x) : x \in \mathbb{R}\} = \langle (2, -3, 2) \rangle.$$

The matrix P *with these eigenvectors as columns is:*

$$P = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 3 & -3 \\ 1 & -2 & 2 \end{pmatrix}.$$

4. *Calculate* P^{-1} *and the diagonal matrix* D*:*

$$\det(P) = -30,$$

$$P^{-1} = \frac{1}{-30}\begin{pmatrix} 0 & -12 & -18 \\ -5 & 0 & 5 \\ -5 & 6 & -1 \end{pmatrix}^T .$$

The diagonal matrix is

$$D = P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

Example 1.5.2 *Let the matrix*

$$B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{C}).$$

1. *Determine the characteristic polynomial of B:*

$$P_B(\lambda) = \det\begin{pmatrix} 1-\lambda & 1 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)(\lambda^2 - 2\lambda + 2).$$

The eigenvalues of B are 1, $1+i$, and $1-i$.

2. *Show that B is diagonalizable: The matrix B is not diagonalizable over $\mathbb{R}$ because its characteristic polynomial does not have all its roots in $\mathbb{R}$. However, it is diagonalizable over $\mathbb{C}$ because it has three distinct eigenvalues.*

3. *Determine a basis of eigenvectors and the change of basis matrix P: For $\lambda = 1$:*

$$\begin{cases} x + y - z = x \\ y = y \\ x + z = z \end{cases} \iff \begin{cases} y = z \\ x = 0 \end{cases}$$

so

$$E_1 = \{(0, y, y) : y \in \mathbb{R}\} = \langle (0,1,1) \rangle.$$

For $\lambda = 1 + i$:

$$\begin{cases} x + y - z = (1+i)x \\ y = (1+i)y \\ x + z = (1+i)z \end{cases} \iff \begin{cases} y = 0 \\ z = -ix \end{cases}$$

so

$$E_{1+i} = \{(x, 0, -ix) : x \in \mathbb{R}\} = \langle (1, 0, -i) \rangle.$$

For $\lambda = 1 - i$*:*

$$\begin{cases} x + y - z = (1-i)x \\ y = (1-i)y \\ x + z = (1-i)z \end{cases} \iff \begin{cases} y = 0 \\ z = ix \end{cases}$$

so

$$E_{1-i} = \{(x, 0, ix) : x \in \mathbb{R}\} = \langle (1, 0, i) \rangle.$$

The matrix P *with these eigenvectors as columns is:*

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -i & i \end{pmatrix}.$$

4. *Calculate* P^{-1} *and the diagonal matrix* D*:*

$$P^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2i} & \frac{1}{2i} \\ \frac{1}{2} & \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix}.$$

The diagonal matrix is

$$D = P^{-1}BP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{pmatrix}.$$

Example 1.5.3 *Let the matrix*

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \in M_4(\mathbb{R}).$$

1. *Determine the characteristic polynomial of* C*:*

$$P_C(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 1 \\ 1 & 1 & 1 & 1-\lambda \end{pmatrix}.$$

Adding the last three columns to the first column gives $4-\lambda$ *in the entire first column. Factor it out and subtract the first row from all others:*

$$P_C(\lambda) = (4-\lambda)\det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix} = \lambda^3(\lambda-4).$$

The eigenvalues of C are the roots of $P_C(\lambda)$*, thus 4 (simple) and 0 (triple).*

2. Show that C is diagonalizable:

We have $\dim E_4 = 1$*. Also,*

$$E_0 = \{(x,y,z,t) \in \mathbb{R}^4 : C\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \mathbf{0}\}.$$

The system is:

$$\begin{cases} x+y+z+t = 0, \\ x+y+z+t = 0, \\ x+y+z+t = 0, \\ x+y+z+t = 0, \end{cases} \iff t = -x-y-z.$$

Thus,

$$E_0 = \{(x,y,z,-x-y-z) : x,y,z \in \mathbb{R}\} = \operatorname{span}\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1)\}.$$

The family $\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1)\}$ *is linearly independent and forms a basis of* E_0*, so* $\dim E_0 = 3$*, equal to the multiplicity of eigenvalue 0. Therefore, C is diagonalizable.*

3. Determine a basis of eigenvectors and matrix P*: The family* $\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1)\}$ *forms a basis of* E_0*. Next, find a basis of* E_4*:*

$$E_4 = \{(x,y,z,t) \in \mathbb{R}^4 : C\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}\}.$$

From the system

$$\begin{cases} -3x + y + z + t = 0, \\ x - 3y + z + t = 0, \\ x + y - 3z + t = 0, \\ x + y + z - 3t = 0, \end{cases} \iff x = y = z = t,$$

we have

$$E_4 = \{(x, x, x, x) : x \in \mathbb{R}\} = \operatorname{span}\{(1, 1, 1, 1)\}.$$

The change of basis matrix is

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

4. Compute $\det P$ *and* P^{-1}*, then diagonal matrix* D*:*

$$\det P = -4 \neq 0,$$

so P *is invertible. Thus,*

$$D = P^{-1}CP = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$