Chapter 01: Properties of the Real Numbers set $\mathbb{R}$

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بالعربية:

- بابا حامد، بن حبيب، التحليل 1 تذكير بالدروس و تمارين محلولة عدد 300 ترجمة الحفيظ مقران، ديوان المطبوعات الجامعية (**الفصل الأول**) .

In English:

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En français:

- BOUHARIS Epouse, OUDJDI DAMERDJI Amel, Cours et exercices corrigés d'Analyse 1, Première année Licence MI Mathématiques et Informatique, U.S.T.O 2020-2021 **(Chapitre 1)**.
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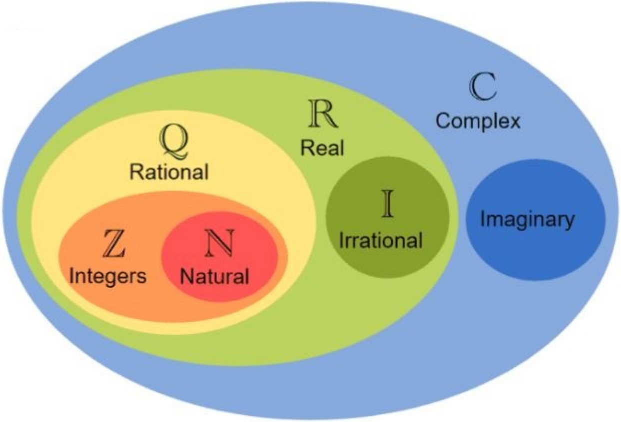
Sets:

- In mathematics, the set is a fundamental concept, which refers to class or collection of objects with specified characteristics.
- For example, we can refer to the set of university students, or the set of letters in the English alphabet.
- The individual objects within a set are known as members or elements.
- Any portion of a set is called a subset of the original set, for example, $\{A, B, C\}$ is a subset of the English alphabet set.
- A set that contains no elements is referred to as the empty set or null set $\emptyset$.

Numbers:



Source: https://owlcation.com/stem/How-to-use-complex-numbers

Real Numbers:

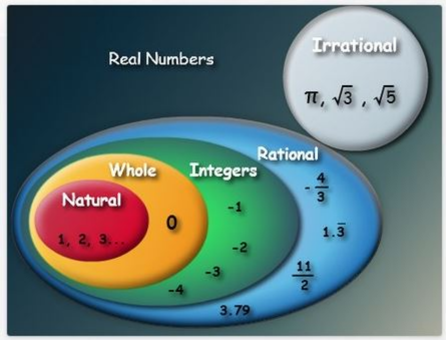


Figure: :
https://empoweryourknowledgeandhappytrivia.wordpress.com/2016/09/06/what
are-the-types-of-real-numbers/

There are different types of numbers already familiar we students:

- Natural numbers:$\{1, 2, 3, \ldots\}$, also known as positive integers, are used for counting the members of a set.
- The sum $a + b$ and product $a \cdot b$ or ab of any two numbers a and b is also a natural number. This is often expressed by stating that the set of natural numbers is closed under the operations of addition and multiplication, or satisfies the closure property with respect to these operators.
- Negative integers and zero, denoted by $\{-1, -2, -3, \ldots\}$ and 0 respectively, were introduced to allaw for solutions of equation such as $x + b = a$ where a and b are any natural numbers. This leads to the operation of subtraction, the inverse of addition, and we write $x = a - b$.
- The set of positive and negative integers and zero is called the set of integers.

- Rational numbers or fractions such as $\frac{4}{3}$ arose to permit solutions of equation such as $bx = a$ for all integers a and b where $a \neq 0$. This leads to the operation of division; the inverse of multiplication, and we write $x = \frac{a}{b}$ or $a \div b$, where a is the numerator and b is the denominator. The set of integers is a subset of the rational numbers, since integers correspond to rational numbers where $b = 1$.
- Irrational numbers such as π and $\sqrt{2}$ are numbers which are not rational, meaning they cannot be expressed as $\frac{a}{b}$ where a and b ere integers and $b \neq 0$.
- The set of rational and irrational numbers is called the set of real numbers.

Operations with Real numbers

When considering numbers a ,b, and c from the set of real numbers $\mathbb{R}$, we have the following operations and properties:

1. Closure under Addition: $a+b$ is a real number.
2. Commutative Law of Addition: $a+b=b+a$.
3. Associative Law of Addition: $a+(b+c)=(a+b)+c$.
4. Additive Identity, call it 0: $a+0=0+a=a$.
5. Addition inverse: For any a there exists x such that $x+a=0$. (denoted by $-a$).
6. Closure under Multiplication: $a \cdot b$ is a real number.
7. Commutation Law of Multiplication: $a \cdot b = b \cdot a$.
8. Associative Law of Multiplication: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
9. Multiplicative Identity, call it 1: $1 \cdot a = a \cdot 1 = a$.
10. Multiplicative Inverse: For $a \neq 0$: for every a, there exists x such that $a \cdot x = 1$ (denoted by $\frac{1}{a}$ or a^{-1}).
11. Distributive Law of Multiplication over addition: $a \cdot (b+c) = a \cdot b + a \cdot c$.

- Generally, any set; like $\mathbb{R}$, whose elements satisfy the above properties is known as a field.
- For example, the set of integers $\mathbb{Z}$ is not a field because it does not satisfy the property of multiplicative inverse (10).
- The field has not the same meaning of the word field, which is usually used in physics.

- **Axiomatic Definition:**
- The set of real numbers is denoted by $\mathbb{R}$, on which two internal composition laws are define:
  - **addition**

$$\begin{aligned} "+" : & \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ & (x, y) \to x + y \end{aligned}$$

  - **multiplication**

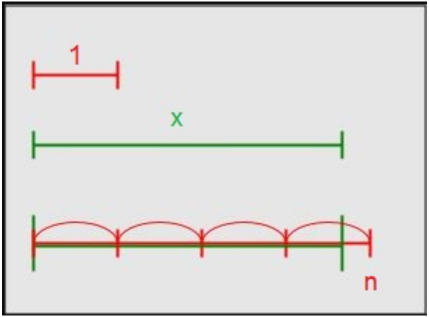
$$\begin{aligned} "\cdot" : & \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ & (x, y) \to x \cdot y \end{aligned}$$

such that $(\mathbb{R}, +, \cdot)$ forms a commutative archimedean field.

- The relation $" \leq "$ is a total order relation on $\mathbb{R} : \forall (x, y) \in \mathbb{R}^2 : (x \leq y) \vee (y \leq x)$.
- The two internal composition laws defined on $\mathbb{R}$ are compatible with the total order relation $" \leq "$.
- Any non-empty and upper-bounded subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R}$.

Archimedean Property

- The field of real numbers $\mathbb{R}$ satisfies **the Archimedean principle**, which is stated as follows:
  $\forall x \in \mathbb{R}^{+} : \exists n \in \mathbb{N} : x < n.$



- This means that $\mathbb{R}$ is **not bounded from above**.

Absolute Value:

- **Definition:** The absolute value is a mapping from $\mathbb{R}$ to the set of positive real numbers $\mathbb{R}^+$, denoted by $|.|$ and defined by: $|.| : \mathbb{R} \to \mathbb{R}^+$

$$|.| : \mathbb{R} \to \mathbb{R}^+$$
$$x \mapsto |x| = \begin{cases} x \text{ } if \text{ } x \geq 0 \\ -x \text{ } if \text{ } x < 0 \end{cases}$$

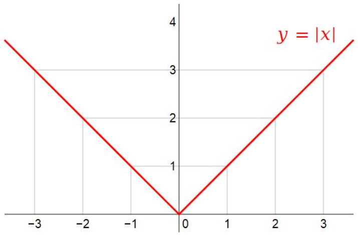


Figure: Source: https://upload.wikimedia.org

- **Properties:**
  1. $|x| \geq 0, \forall x \in \mathbb{R}.$
  2. $|x| = 0 \Leftrightarrow x = 0.$
  3. $-|x| \leq x \leq |x|; \forall x \in \mathbb{R}.$
  4. $\forall a \geq 0; |x| \leq a \Leftrightarrow -a \leq x \leq a.$
  5. $|x \cdot y| = |x| \cdot |y|, \forall x, y \in \mathbb{R}.$
  6. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, \forall (x, y) \in \mathbb{R} \times \mathbb{R}^*.$
  7. $|x + y| \leq |x| + |y|, \forall x, y \in \mathbb{R},$ (The triangular inequality).
  8. $||x| - |y|| \leq |x - y|, \forall x, y \in \mathbb{R},$ (The second triangular inequality)

Proof: (of the two last properties)

- $\forall x, y \in \mathbb{R}$

$$\begin{cases} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{cases} \quad (1)$$

Hence, by adding them up

$$-(|x| + |y|) \leq x + y \leq |x| + |y| \Leftrightarrow |x + y| \leq |x| + |y|.$$

- $\forall x, y \in \mathbb{R}$

$$|x| = |x - y + y| \Rightarrow |x| \leq |x - y| + |y| \Leftrightarrow |x| - |y| \leq |x - y|$$

and

$$|y| = |y - x + x| \Rightarrow |y| \leq |y - x| + |x| \Leftrightarrow -|x - y| \leq |x| - |y|$$

Therefore,

$$-|x - y| \leq |x| - |y| \leq |x - y| \Leftrightarrow ||x| - |y|| \leq |x - y|. \square$$

Definition: The floor function of a real number x is **the largest integer n that is less than or equal to x**. In other words, the floor of x is the unique integer n in $\mathbb{Z}$ such that $n \leq x < n+1$. It is denoted as $[x]$ or $E(x)$. Thus, every real number x can be uniquely expressed as $x = [x] + \alpha$, where $\alpha \in [0, 1[$.

Example: $[4.60811] = 4$, $[-4.60811] = -5$

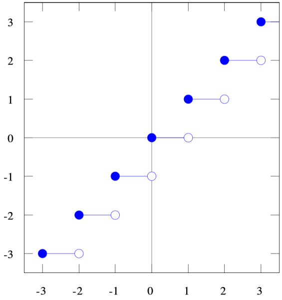


Figure: Source:https://upload.wikimedia.org

Properties:

- $[x] \in \mathbb{Z}$, for all $x \in \mathbb{R}$.
- $[x] \leq x \leq [x] + 1$, for all $x \in \mathbb{R}$.
- $[x + m] = [x] + m$, for all $x \in \mathbb{R}$ and $m \in \mathbb{Z}$.
- $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$, for all $x, y \in \mathbb{R}$.
- $x \leq y \Rightarrow [x] \leq [y]$, for all $x, y \in \mathbb{R}$.

Proof:

- For any $x \in \mathbb{R}$, we have

$$[x] \leq x \leq [x] + 1,$$

which implies

$$[x] + m \leq x + m \leq [x] + m + 1, \tag{2}$$

for all $m \in \mathbb{Z}$. On the other hand,

$$[x + m] \leq x + m \leq [x + m] + 1, \tag{3}$$

since $[x + m]$ is the largest integer less then $x + m$, we have

$$[x] + m \leq [x + m] \tag{4}$$

Similarly, $[x + m] + 1$ is the smallest integer greater than or equal to $x + m$, so

$$[x+m]+1 \leq [x]+m+1 \tag{5}$$

Combining these, we get

$$[x+m] \leq [x]+m \tag{6}$$

From $[x]+m \leq [x+m]$ and $[x+m] \leq [x]+m$, we conclude $[x+m] = [x]+m$. □

Note: The floor function is increasing function but not strictly increasing.

Definition: A subset I of $\mathbb{R}$ is an interval in $\mathbb{R}$, if whenever it contains two real numbers a and b, so that it contains all real numbers lying between them.
$\forall a, b \in I, \forall x \in \mathbb{R}; a \leq x \leq b \Rightarrow x \in I$.

Examples:

- $\mathbb{R}$ and the empty set $\emptyset$ are intervals.
- $\mathbb{R}^+$ is an interval.
- $\mathbb{R}^*$ and $\mathbb{N}$ are not intervals.

Remarks:

- For the notations, let a, $b \in \mathbb{R}$, we have the intervals in $\mathbb{R}$:
  - Bounded: open $]a, b[$, closed $[a, b]$, or half-open intervals $[a, b[,]a, b]$.
  - Unbounded: open $]-\infty, b[$, $]a, +\infty[$, or closed $[a, +\infty[$, $]-\infty, b]$.
  - If $a = b$, then $[a, a] = \{a\}$, $]a, b[= [a, b[=]a, b] = \emptyset$.

- The complement of an open interval is closed.

Note: $\mathbb{R}$ and the empty set $\emptyset$ are the only open and closed sets of $\mathbb{R}$.

Indeed, $\mathbb{R} =]-\infty, +\infty[$ is an open interval, so its complement, the empty set $\emptyset$ is closed. Moreover, the empty set $\emptyset$ can be written as an open interval $]\alpha, \alpha[$, where $\alpha \in \mathbb{R}$, so its complement $\mathbb{R}$ is closed.

Remarks:

- The intersection of two intervals is always an interval.
- The union of two intervals with a non-empty intersection is an interval.

Definition: Let $a, b \in \mathbb{R}$. We define the segment denoted as $[a, b]$ by $[a, b] = \{x \in \mathbb{R}/a \leq x \leq b\}$. If $a > b$ then $[a, b] = \emptyset$.
Definition: Let V be a subset of $\mathbb{R}$ and $x_0 \in \mathbb{R}$. We say that V is a neighborhood of x_0 if there exists an open interval $]a, b[$ in $\mathbb{R}$ containing x_0 and contained in V. We denoted this as V_{x_0} or $V(x_0)$.
Examples:

- For any $\varepsilon > 0$, the interval $V =]x_0 - \varepsilon, x_0 + \varepsilon[$ is a neighborhood of x_0 because there exists an open interval $]x_0 - \frac{\varepsilon}{2}, x_0 + \frac{\varepsilon}{2}[$ in $\mathbb{R}$ containing x_0 and contained V.
- The interval $]a, b[$ is a neighborhood of all points $x \in]a, b[$.
- The sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are not neighborhoods of any of their points.

Lower bounds, upper bounds, infimum, supremum, Maximum, Minimum

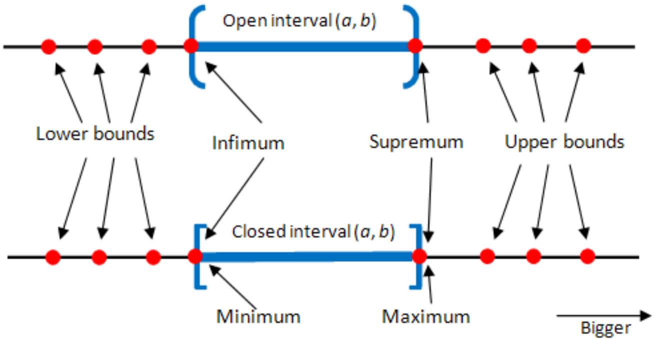


Figure: Source:https://mathstrek.blog/2012/11/09/basic-analysis-sequence-convergence-1/

Definition: Given a subset $E \subset \mathbb{R}$ totally ordered by the relation denoted as $" \leq "$, and let $A \subset E$ be a non-empty subset of E.

- We say that $M \in E$ is an upper bound of A if: $\forall x \in A, x \leq M$.
- We say that $m \in E$ is lower bound of A if: $\forall x \in A, m \leq x$.
- A is said to be bounded above (respectively, bounded below) if it has at least one upper bound (respectively, lower bound).

Note: If A has an upper bound (respectively, lower bound), it is not necessarily unique.

Definition:

- Given a non-empty and bounded above subset A of E, and let $Maj(A) \subset E$ be set of upper bounds A, we say that $M \in E$ is the supremum of A if M is the smallest of the upper bounds of A, and we denote it as $\sup A$.
- Given a non-empty and bounded below subset A of E, and let $Min(A) \subset E$ be the set of lower bounds of A, we say that $m \in E$ is the infimum of A if m is the largest lower bounds of A, and we denote it as $\inf A$.

Theorem: Every non-empty and bounded above (respectively, bounded below) subset of $\mathbb{R}$ has a supremum (respectively, infimum.)

Remarks:

- When the supremum (respectively, infimum) exists, it is unique.
- The supremum $\sup A$ (repectively, infimum $\inf(A)$) does not necessarily belong to the set A.

Definition:

- We say that M is the greatest element of A or the maximum of A if M is an upper bound of A that belongs to A, and we denote it as $\max A$.
- We say that m is the smallest element of A that belongs to A, and we denote it as $\min A$.

Remarks:

- If the maximum $\max A$ (respectively, minimum $\min A$) exists, then $\sup A = \max A$ (respectively $\inf A = \min A$).
- If the supremum $\sup A$ (respectively , infimum $\inf A$) belongs to A, then $\max A = \sup A$ (respectively, $\min A = \inf A$).
- If the supremum $\sup A$ (respectively, infimum $\inf A$) does not belong to A, then the maximum $\max A$ (respectively, minimum $\min A$) does not exist.

Note: The superemum of a bounded above set A (respectively, the infimum of a bounded below set A) always exist but may not belong to A. On the other hand, the maximum of a bounded above set (respectively, the minimum of a bounded below set) may not exist.

Example: Let $A =]-7, 2]$; A is a bounded subset of $\mathbb{R}$. The set of upper bounds of A is $Maj(A) = [2, +\infty[$ with $\sup A = \max A = 2$. The set of lower bounds of A is $Min(A) =]-\infty, -7]$, with $\inf A = -7$; however, $\min A$ does not exist because $-7 \notin A$.

Proposition: Let A be non-empty subset of $\mathbb{R}$. The following two statements are equivalent:

- (i) $\exists \alpha > 0, \forall x \in A : |x| \leq \alpha$.
- (ii) $\exists m, M \in \mathbb{R}, \forall x \in A : m \leq x \leq M$.

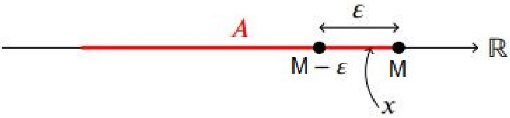
Proof:

- $(i) \Rightarrow (ii)$ it suffices to take $m = -\alpha$ and $M = \alpha$.
- $(ii) \Rightarrow (i)$ it suffices to take $\alpha = \max(M, -m)$, indeed, $-\alpha \leq m \leq x \leq M \leq \alpha \Rightarrow -\alpha \leq x \leq \alpha \iff |x| \leq \alpha$.

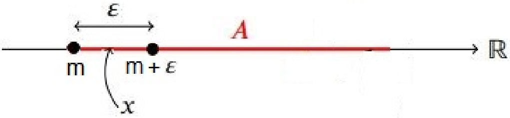
Characterization of the supremum and of the infimum:

Given a non-empty and bounded set A in $\mathbb{R}$, let m and M be real numbers. We have the following characterizations:

1. $M = \sup A$ if and only if:
   - For all $x \in A$; $x \leq M$.
   - For all $\epsilon > 0$; there exists $x \in A$ such that $M - \epsilon < x$.



2. $m = \inf A$ if and only if:
   - For all $x \in A$; $m \leq x$.
   - For all $\epsilon > 0$; there exists $x \in A$ such that $x < m + \varepsilon$.



Proof:

1. - First, let's show that if $M = \sup A$, then for any $\epsilon > 0$, there exists $x \in A$ such that $M - \epsilon < x$.
     We assume by contradiction that there exists $\epsilon > 0$ such that for all $x \in A$; $x \leq M - \epsilon$. Consequently, $M - \epsilon$ becomes an upper bound for A. However, since M is least upper bound of A, we have: $M \leq M - \epsilon \iff \epsilon \leq 0$, which is a contradiction.
   - Now let's show that if M is an upper bound of A and satisfies $\forall \epsilon > 0; \exists x_0 \in A, M - \epsilon < x_0$, then M is the least upper bound of A. Let M_0 be another upper bound of A, hence $x_0 \leq M_0$.
     Therefore: $\forall \epsilon > 0; M - \epsilon < x_0 \leq M_0 \Rightarrow \forall \epsilon > 0; M - M_0 < \epsilon$, which implies $M - M_0 < \epsilon$, which implies
     $M - M_0 \leq 0 \iff M \leq M_0$.
2. The characterization of the lower bound can be shown in a similar manner.$\square$

Properties:

1. Given two non-empty, bounded sets A and B in $\mathbb{R}$, such that $A \subset B$, then:

$$\inf B \leq \inf A \leq \sup A \leq \sup B. \tag{7}$$

In fact, we have

- $\inf A \leq x \leq \sup A$ for all $x \in A$, which implies $\inf A \leq \sup A$.
- For every x in A, we also have $x \in B$, which leads to $\inf B \leq x$ for all x in A.

This means $\inf B$ is a lower bound for A, but $\inf A$ is the greatest lower bound for A, so $\inf B \leq \inf A$. Furthermore,

- for all x in A (and hence x in B), we have $x \leq \sup B$,

which means $\sup B$ is an upper bound for A. But $\sup A$ is the smallest upper bound for A, so $\sup A \leq \sup B$.

2. Given C and D, two non-empty, bounded sets in $\mathbb{R}$, then:
   - $\sup(C \cup D) = \max(\sup C, \sup D)$
     $\inf(C \cup D) = \min(\inf C, \inf D)$
   - $\sup(C \cap D) \leq \min(\sup C, \sup D)$
     $\inf(C \cap D) \geq \max(\inf C, \inf D)$
   - $\sup(C + D) = \sup C + \sup D$
     $\inf(C + D) = \inf C + \inf D$
     where $C + D = \{x + y / x \in C, y = D\}$

Theorem: Given two distinct real numbers a and b such that $a < b$, then the interval $]a, b[$ contains at least one rational number $q \in \mathbb{Q}$. We say that $\mathbb{Q}$ is dense in $\mathbb{R}$ and denote it as $\bar{\mathbb{Q}} = \mathbb{R}$.

Thanks