Chapter 1

Real functions of several real variables

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In general, the real function of several real variables are of the form

$$y = f(x_1, x_2, \cdots, x_n)$$

where $x_1, x_2, \cdots, x_n$ and y are real numbers.

1.1 Functions of two variables

Definition 1.1.1.

We call a function of two variables a function f from $\mathbb{R}^2$ into $\mathbb{R}$

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x,y) & \longmapsto & f(x,y) \end{array}$$

Definition 1.1.2.

We call the domain of definition of f denoted D_f the set of elements of $\mathbb{R}^2$ which have an image by f

$$D_f = \{(x,y) \in \mathbb{R}^2 \ \ f(x,y) \ \ \textit{is defined} \ \}$$

Example 1.1.1. *The function $f(x,y) = \sqrt{1 - x^2 - y^2}$ is a function of two variables whose D_f is the disk with center (0.0) and radius* 1

$$D_f = \{(x,y) \in \mathbb{R}^2 \ \ , x^2 + y^2 \leq 1\}$$

1.1.1 Graphic representation

Let f be a function of two variables. We call graph of f a part of $\mathbb{R}^2 \times \mathbb{R}$ such that

$$G_f = \left\{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y), (x,y) \in D\right\}$$

G_f is calledé surface area.

1.1.2 Partial derivatives of order 01

Let f be a function of two variables defined on a part D of $\mathbb{R}^2$ and Let $(x, y), (x_0, y_0)$ be two vectors of $\mathbb{R}^2$.

When we fix one of the two variables we obtain a real function of a single real variable.

$\longrightarrow$ If we fix y (let's say $y = y_0$) we can study f as a function of the single variable x i.e. $f(x, y_0) = f_1(x)$ and we can calculate its derivative at x_0 when this limit exists

$$\begin{aligned} f_1'(x_0) &= \lim_{x \longrightarrow x_0} \frac{f_1(x) - f_1(x_0)}{x - x_0} \\ &= \lim_{h \longrightarrow 0} \frac{f_1(x_0 + h) - f_1(x_0)}{h} \\ &= \lim_{h \longrightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \end{aligned}$$

$\longrightarrow$ If we fix x (let's say $x = x_0$) we can study f as a function of the single variable y i.e. $f(x_0, y) = f_2(y)$ and we can calculate its derivative at y_0 when this limit exists

$$\begin{aligned} f_2'(y_0) &= \lim_{y \longrightarrow y_0} \frac{f_2(y) - f_2(y_0)}{y - y_0} \\ &= \lim_{h \longrightarrow 0} \frac{f_2(y_0 + h) - f_2(y_0)}{h} \\ &= \lim_{h \longrightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \end{aligned}$$

Definition 1.1.3.

⟶ *We call the partial derivative of f at the point (x_0, y_0) with respect to the variable x the real $f_1'(x_0)$ and we denote it $f_x'(x_0, y_0)$ or $\frac{\partial f}{\partial x}(x_0, y_0)$*

$$f_x'(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \longrightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

⟶ *We call the partial derivative of f at the point (x_0, y_0) with respect to the variable y the real $f_2'(y_0)$ and we denote it $f_y'(x_0, y_0)$ or $\frac{\partial f}{\partial y}(x_0, y_0)$*

$$f_y'(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{h \longrightarrow 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

1.1.3 Gradient

Definition 1.1.4.

If the function f admits partial derivatives of order 01 at the point (x_0, y_0), the vector $gradf(x_0, y_0)$ defined by

$$gradf(x_0, y_0) = \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right)$$

and we denote it by $\nabla f(x_0, y_0)$

1.1.4 Partial derivatives of order 02

Definition 1.1.5.

Under condition of existence, we call partial derivatives of order 02 of f at the point (x_0, y_0) the partial derivatives of the functions $f'_x : \ (x, y) \longrightarrow f'_x(x, y)$ and $f'_y : \ (x, y) \longrightarrow f'_y(x, y)$

We will therefore have four derivatives of order 02

$$f''_{x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]$$

$$f''_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]$$

$$f''_{y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]$$

$$f''_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right]$$

Example 1.1.2. *Partial derivatives of order 01 and 02 of the function $f(x, y) = x^2 + y^2 + 3xy$*

$$\frac{\partial f}{\partial x}(x, y) = 2x + 3y \qquad \frac{\partial f}{\partial y}(x, y) = 2y + 3x$$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 2 \qquad \frac{\partial^2 f}{\partial y^2}(x, y) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = 3 \qquad \frac{\partial^2 f}{\partial y \partial x}(x, y) = 3$$

1.1.5 Differentials

Let f be a function of two variables and $M_0(x_0, y_0)$ be a point of $\mathbb{R}^2$, the map

$$\begin{array}{rccc} u: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (h_1, h_2) & \longmapsto & h_1 \frac{\partial f}{\partial x}(M_0) + h_2 \frac{\partial f}{\partial y}(M_0) \end{array}$$

is linear of $\mathbb{R}^2$ into $\mathbb{R}$ i.e.

$$u(p+q) = u(p) + u(q) \quad \forall p, q \in \mathbb{R}^2$$

$$u(\lambda p) = \lambda u(p) \quad \forall p \in \mathbb{R}^2, \ \forall \lambda \in \mathbb{R}$$

The map u is said to be differential from f to M_0 and we denote it $df(M_0)$

Theorem 1.1.1. *Let f be a function defined in the neighborhood of $M_0 \in \mathbb{R}^2$ and admitting continuous partial derivatives in the neighborhood of M_0, then f is differentiable in M_0 and*

$$df(M_0) = \frac{\partial f}{\partial x}(M_0)dx + \frac{\partial f}{\partial y}(M_0)dy$$

1.2 Double integral

The double integral is the generalization of a simple integral, i.e. the double integral is calculated by making two successive integrations denoted $\iint_D f(x,y)dxdy$ where f is a continuous function on a finite domain D of the plane $\mathbb{R}^2$.

1.2.1 Integration on a rectangle

Let $D = [a,b] \times [c,d]$ be a rectangle of $\mathbb{R}^2$ and let f be a continuous function on D with real values, then

$$\iint_D f(x,y)dxdy = \int_a^b \Big[\int_c^d f(x,y)dy \Big] dx$$

and according to Fubini's theorem, we can also write

$$\iint_D f(x,y)dxdy = \int_a^b \Big[\int_c^d f(x,y)dy \Big] dx = \int_c^d \Big[\int_a^b f(x,y)dx \Big] dy$$

Example 1.2.1. *Calculate the following double integral:*

$$I = \iint_D 2x\, dx\, dy \qquad D = [-1, 2] \times [-1, 1]$$

$$\begin{aligned} I &= \iint_D 2x\, dx\, dy = \int_{-1}^{1} \Big[\int_{-1}^{2} 2x\, dx \Big] dy \\ &= \int_{-1}^{1} \big[x^2\big]_{-1}^{2} dy \\ &= \int_{-1}^{1} 3\, dy = 6 \end{aligned}$$

Remark 1.2.1. *If $f(x, y) = g(x)h(y)$ where $g : [a, b] \longrightarrow \mathbb{R}$ and $h : [c, d] \longrightarrow \mathbb{R}$ are continuous functions, then*

$$\int_a^b \int_c^d f(x, y) dx dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

1.2.2 Integration on a non-rectangular domain

If the domain of integration D is of the form

$$D = \{(x, y) \in \mathbb{R}^2 |\ a \le x \le b \text{ and } y_1(x) \le y \le y_2(x)\}$$

Then

$$\iint_D f(x, y) dx dy = \int_{x=a}^{x=b} \Big[\int_{y_1(x)}^{y_2(x)} f(x, y) dy \Big] dx$$

The general method of calculating $\iint_D f(x, y) dx dy$ consists of first integrating with respect to a variable, y for example, the limits depending on x then to integrating with respect to the other variable.

Example 1.2.2. *Calculate the following double integral:*

$I = \iint_D xy\, dx\, dy \qquad$ *where* $\qquad D = \{(x, y) \in \mathbb{R}^2 |\ x, y \ge 0,\ x + y \le 1\}$

We have

$$D = \{(x,y) \in \mathbb{R}^2 |\ 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1-x\}$$

Then

$$\begin{aligned} I &= \iint_D xy\ dx\ dy = \int_0^1 \Big[\int_0^{1-x} xy\ dy \Big] dx \\ &= \int_0^1 \Big[\frac{1}{2}xy^2\Big]_0^{1-x} dx \\ &= \frac{1}{2}\int_0^1 x(1-x)^2\ dx \\ &= \frac{1}{2}\int_0^1 x^3 - 2x^2 + x\ dx \\ &= \frac{1}{24} \end{aligned}$$

1.3 Triple integral

The principle of the triple integral is the same as for the double integral, just replacing a small surface element with a small volume element.

1.3.1 Fubini's theorem on a parallelepiped

Theorem 1.3.1. *Let f be a continuous function on a parallelepiped $P = [a,b] \times [c,d] \times [e,f]$, then we have*

$$\begin{aligned} \iiint_P f(x,y,z) dxdydz &= \int_a^b \Big[\int_c^d \int_e^f f(x,y,z) dzdy \Big] dx \\ &= \int_c^d \Big[\int_a^b \int_e^f f(x,y,z) dzdx \Big] dy \\ &= \int_e^f \Big[\int_a^b \int_c^d f(x,y,z) dydx \Big] dz \end{aligned}$$

Example 1.3.1. *Calculate* $I = \int_0^1 \int_0^1 \int_0^1 2\Big(xy + yz + zx\Big)dxdydz$

$$\begin{aligned} I &= \int_0^1 \int_0^1 \Big[\int_0^1 2\Big(xy + yz + zx\Big)dx\Big]dydz \\ &= \int_0^1 \int_0^1 \Big[x^2y + 2yzx + zx^2\Big]_0^1 dydz \\ &= \int_0^1 \int_0^1 y + 2yz + zdydz \\ &= \int_0^1 \Big[\frac{y^2}{2} + y^2z + zy\Big]_0^1 dz \\ &= \int_0^1 \frac{1}{2} + 2zdz \\ &= \frac{3}{2}. \end{aligned}$$

1.3.2 Fubini's theorem on a domain P of $\mathbb{R}^3$

The idea is to take one of the three variables x, y, z varies between two extreme limits a and b let us suppose for example z therefore the plane domain obtained by cutting the volume P by a plane $z = constant$ is a simple domain so that we can calculate the double integral $\iiint_D f(x, y, z)dxdy$ and we have

$$\iiint_P f(x, y, z)dxdydz = \int_a^b \Big[\iint_D f(x, y, z)dxdy\Big]dz$$

Example 1.3.2. *Calculate the following integral:*

$$I = \iint_P dx\,dy\,dz \;\; où \;\; P = \{(x, y, z) \in \mathbb{R}^3 |\;\; x, y, z \geq 0,\;\; x + y + 2z \leq 1\}$$

It is therefore a question of calculating the volume of P*, we cut* P *by a horizontal plane* $z = z_0$ *we then find a triangle* D *according to* x *and* y *limited by* $x = 0, y = 0$ *and*

$x + y = 1 - 2z_0$ *such that* $z_0 \in [0, \frac{1}{2}]$ *and therefore*

$$\begin{aligned} 0 &\leq z \leq \frac{1}{2} \\ 0 &\leq y \leq 1 - 2z - x \\ 0 &\leq x \leq 1 - 2z \end{aligned}$$

Then

$$\begin{aligned} I &= \iiint_P dx\ dy\ dz \\ &= \int_0^{\frac{1}{2}} \int_0^{1-2z} \int_0^{1-2z-x} dydxdz \\ &= \int_0^{\frac{1}{2}} \int_0^{1-2z} 1 - 2z - x\, dxdz \\ &= \int_0^{\frac{1}{2}} 2z^2 - 2z + \frac{1}{2}\, dz \\ &= \frac{1}{12} \end{aligned}$$