

corrigé de l'examen (Analyse 2)

Exercice 1:

1) Montre que $\ln(1+n) < n$ (3pts)

$$f(n) = \ln(1+n) = f(0) + \frac{f'(0)}{1!} n + \frac{f''(\theta n)}{2!} n^2$$

$$\text{on a: } f(0) = 0, f'(0) = 1, \text{ et } f''(n) = \frac{-1}{(1+n)^2}, \quad 0 < \theta < 1$$

$$\Rightarrow f''(\theta n) = \frac{-1}{(1+\theta n)^2}$$

$$f(n) = \ln(1+n) = n - \frac{1}{2(1+\theta n)^2} \cdot n^2, \quad 0 < \theta < 1.$$

$$\text{on a: } \frac{-1}{2(1+\theta n)^2} n^2 < 0 \quad (\text{car } n > 0, \text{ et } 0 < \theta < 1).$$

$$\Rightarrow n - \frac{1}{2(1+\theta n)^2} n^2 < n$$

$$\text{alors: } \ln(1+n) < n$$

2) a) $f(n) = e^{\cos n}$, $n_0 = 0$, $h = 4$. (2pts)

$$\text{on a: } \cos n = 1 - \frac{1}{2} n^2 + \frac{1}{24} n^4 + o(n^4)$$

$$\text{donc: } f(n) = e^{\cos n} = e^{1 - \frac{1}{2} n^2 + \frac{1}{24} n^4 + o(n^4)}$$

$$= e \cdot e^{-\frac{1}{2} n^2 + \frac{1}{24} n^4 + o(n^4)}$$

$$\text{on pose: } z = -\frac{1}{2} n^2 + \frac{1}{24} n^4 \quad (\text{si } n \rightarrow 0 \Rightarrow z \rightarrow 0)$$

$$e^z = 1 + z + \frac{z^2}{2!} + o(z^2)$$

$$= 1 + \left(-\frac{x^2}{2} + \frac{x^4}{24} \right) + \frac{1}{2!} \left(-\frac{x^2}{2} + \frac{x^4}{24} \right)^2 + o(x^4)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^4}{8} + o(x^4) = 1 - \frac{x^2}{2} + \frac{x^4}{6} + o(x^4)$$

donc : $f(x) = e^{\cos x} = e \left(1 - \frac{x^2}{2} + \frac{x^4}{6} + o(x^4) \right)$

$$= e - \frac{ex^2}{2} + \frac{ex^4}{6} + o(x^4)$$

b) $g(x) = \frac{x^3}{x-1}$, $x_0 = \infty$, $n=3$. (2pts)

suppose que : $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$ (si $x \rightarrow +\infty \Rightarrow y \rightarrow 0$)

$$f(y) = \frac{\left(\frac{1}{y}\right)^3}{\left(\frac{1}{y}\right) - 1} = \frac{\frac{1}{y^3}}{\frac{1-y}{y}} = \frac{1}{y^2} \left(\frac{1}{1-y} \right)$$

$$= \frac{1}{y^2} (1 + y + y^2 + y^3 + y^4 + y^5 + o(y^5))$$

$$= \frac{1}{y^2} + \frac{1}{y} + 1 + y + y^2 + y^3 + o(y^3)$$

donc : $g(x) = 1 + x^2 + x + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + o(x^3)$

3) a) Montre que : $\text{Arctan } x > \frac{x}{1+x^2} \quad \forall x > 0$. (1,5pts)

soit la fonction $f(x) = \text{Arctan } x - \frac{x}{1+x^2}$

f est dérivable et $f'(x) = \frac{2x^2}{(1+x^2)^2} \geq 0$. donc f est croissante.

ona : $x > 0 \Rightarrow f(x) > f(0) = 0$

$$\Rightarrow \text{Arctan } x - \frac{x}{1+x^2} > 0 \Rightarrow \text{Arctan } x > \frac{x}{1+x^2}, \quad \forall x > 0$$

(2)

b) $\text{Arcsin } x + \text{Arccos } x = \frac{\pi}{2}$, $\forall x \in]-1, 1[$. (1,5 pts)
soit la fonction :

$f(x) = \text{Arcsin } x + \text{Arccos } x$, $f(x)$ est dérivable

sur $]-1, 1[$, on a : $f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$.

donc f est constante sur $]-1, 1[$.

(i.e. : $\forall x \in]-1, 1[$, $f(x) = c$).

on a : $f(0) = \text{Arcsin } 0 + \text{Arccos } 0 = \frac{\pi}{2}$.

Donc : $\forall x \in]-1, 1[$: $\text{Arcsin } x + \text{Arccos } x = \frac{\pi}{2}$.

Exercice 2 :

1) calculons les intégrales : (3 pts)

$$I_1 = \int \frac{1}{5-3\cos x} dx.$$

on pose : $t = \tan \frac{x}{2} \Rightarrow dx = \frac{2 dt}{1+t^2}$ et $\cos x = \frac{1-t^2}{1+t^2}$.

$$I_1 = \int \frac{\frac{2}{1+t^2}}{5+3\left(\frac{1-t^2}{1+t^2}\right)} dt = \int \frac{2}{5(1+t^2)-3(1-t^2)} dt.$$

$$= \int \frac{2}{8t^2+2} dt = \frac{1}{2} \int \frac{1}{1+(2t)^2} dt.$$

on pose $y = 2t \Rightarrow dy = 2 dt$.

$$I_1 = \frac{1}{2} \int \frac{1}{1+y^2} dy = \frac{1}{2} \arctan y + C$$

[3]

donc $I_2 = \frac{1}{2} \arctan\left(2 \tan \frac{x}{2}\right) + \text{C.O.I.}$

$$I_2 = \int_1^2 \frac{\ln(1+n)}{n^2} dn \quad (I. p. p.) (2 \text{ pts})$$

$$\begin{cases} u = \ln(1+n) \\ u' = \frac{1}{1+n} \end{cases} \Rightarrow \text{O.I.V.} \quad \begin{cases} v' = \frac{1}{n^2} \\ v = -\frac{1}{n} \end{cases}$$

$$\Rightarrow I_2 = -\frac{\ln(1+n)}{n} \Big|_1^2 + \int_1^2 \frac{1}{n(1+n)} dn \quad \text{O.I.V.}$$

$$\text{On a: } \frac{1}{n(1+n)} = \frac{1}{n} - \frac{1}{1+n} \quad \text{O.I.V.}$$

$$\begin{aligned} I_2 &= -\frac{\ln(1+n)}{n} \Big|_1^2 + \ln(n) \Big|_1^2 - \ln(1+n) \Big|_1^2 \\ &= -\frac{3 \ln(3)}{2} + 3 \ln(2) \quad \text{O.I.V.} \end{aligned}$$

2) a) $y' = e^{x+y}$ -- (E.S.) (2 pts)

$$\Rightarrow \frac{dy}{dx} = e^x \cdot e^y \Rightarrow e^{-y} \frac{dy}{dx} = e^x$$

$$\Rightarrow e^{-y} dy = e^x dx \Rightarrow \int e^{-y} dy = \int e^x dx \quad \text{O.I.V.}$$

$$\Rightarrow -e^{-y} = e^x + C \Rightarrow e^{-y} = -e^x - C \quad C \in \mathbb{R}$$

$$\Rightarrow y = -\ln(-e^x - C), \quad C \in \mathbb{R} \quad \text{O.I.V.}$$

$$b) \quad y'' + y' + y = x^2 + x + 1 \quad \dots (E_2) \quad (3 \text{ pts})$$

$$y_h? \quad y'' + y' + y = 0. \quad \text{on pose } y = e^{rx}$$

$$\text{donc: } y' = r e^{rx} \quad \text{et} \quad y'' = r^2 e^{rx}.$$

$$\Rightarrow (r^2 + r + 1) e^{rx} = 0 \quad \Rightarrow r^2 + r + 1 = 0 \quad (0.11)$$

$$\Delta = -3 < 0 \quad \Rightarrow \quad r_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i, \quad r_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$\text{donc } y_h = e^{-\frac{1}{2}x} \left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x \right) \quad (0.11)$$

$$c_1, c_2 \in \mathbb{R}.$$

La solution particulière: y_p .

$$y_p = ax^2 + bx + c \Rightarrow y_p' = 2ax + b, \quad y_p'' = 2a.$$

remplaçons y, y', y'' dans (E_2) :

$$2a + 2ax + b + a x^2 + b x + c = x^2 + x + 1.$$

$$\begin{cases} a = 1 \\ 2a + b = 1 \\ 2a + b + c = 1 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = -1 \\ c = 0 \end{cases} \quad (0.11)$$

$$\text{A la fin: } y_p = x^2 - x \quad (0.11)$$

La solution générale de (E_2) :

$$y = y_h + y_p$$

$$= e^{-\frac{1}{2}x} \left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x \right) + x^2 - x$$

$$\text{avec } c_1, c_2 \in \mathbb{R}$$

(5)