

Exercice 1:

Montrer que:

$$\text{Arcsin } x < \frac{x}{\sqrt{1-x^2}}, \quad \forall x \in]0, 1[;$$

$$\text{Arctan } x > \frac{x}{1+x^2}, \quad \forall x > 0;$$

$$\text{Arcsin } x + \text{Arccos } x = \frac{\pi}{2}, \quad \forall x \in]-1, 1[;$$

$$\text{Arctan } x + \text{Arctan } \frac{1}{x} = \frac{\pi}{2}, \quad \forall x > 0.$$

Exercice 2:

1. Les réels x et y étant liés par: $x = \ln \left(\tan \left(\frac{y}{2} + \frac{\pi}{4} \right) \right)$.

- Calculer chx , shx et thx en fonction de y .

2. Calculer les limites suivantes:

$$\lim_{x \rightarrow +\infty} e^{-x} (ch^3 x - sh^3 x) \quad \text{et} \quad \lim_{x \rightarrow +\infty} (x - \ln(chx)).$$

Exercice 3:

En utilisant le développement de Maclaurin de la fonction $\ln(x+1)$, montrer que:

$$\forall x > 0, \quad x - \frac{x^2}{2} < \ln(x+1) < x.$$

Exercice 4:

En utilisant le développement de Taylor-Young, calcule la limite suivante:

$$\lim_{x \rightarrow 0} \frac{\ln(x+1) - x + \frac{x^2}{4}}{(\sin x)^2}.$$

Exercice 5:

1) - Écrire la formule de Maclaurin avec le reste de Lagrange pour la fonction $f(x) = e^x$ à l'ordre n .

- Montrer que: $1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{e}{(n+1)!}$.

- En déduire la limite de la suite $u_n = \sum_{k=0}^n \frac{1}{k!}$.

Exercice 6:

En utilisant la formule de Maclaurin pour la fonction $\ln(x+1)$. Calculer:

$$\lim_{n \rightarrow +\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n-1}}{n} \right).$$

Série 4
Exo 3 - part 2
 $y_n = \ln(1+x)$
dérivée n -ième

Exercice 7:

Trouver la limite de la suite (u_n) définie par:

$$u_n = \prod_{k=1}^n \left(1 + \frac{k}{n^2}\right), \quad \forall n \in \mathbb{N}^*.$$

Exercice 8:

Calculer les développements limités jusqu'à l'ordre n au voisinage de x_0 des fonctions suivantes:

- 1) $f(x) = x(\ln x)^{\frac{1}{2}}, \quad x_0 = 0, \quad n = 3,$
- 2) $f(x) = \ln(1 + \sin x), \quad x_0 = 0, \quad n = 3,$
- 3) $f(x) = \frac{1+x+x^2}{1-x+x^2}, \quad x_0 = 0, \quad n = 4,$
- 4) $f(x) = \cos(\sin x); \quad x_0 = 0, \quad n = 4.$
- 5) $f(x) = \frac{x}{1-x^2}, \quad x_0 = 0, \quad n = 4,$
- 6) $f(x) = \ln \frac{\sin x}{x}, \quad x_0 = 0, \quad n = 4,$
- 7) $f(x) = \frac{1+2x+x^3}{x^3+x^5}, \quad x_0 = 0, \quad n = 2,$
- 8) $f(x) = \exp(\sqrt{x}), \quad x_0 = 1, \quad n = 3,$
- 9) $f(x) = \frac{x^3+2}{x-1}, \quad x_0 = +\infty, \quad n = 3.$

Exercice 9:

Soient f et g deux fonctions qui admettent des développements limités au voisinage de zéro donnés par:

$$f(x) = x - \frac{x^3}{6} + o(x^4); \quad g(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4).$$

- Calculer le développement limité jusqu'à l'ordre 4 au voisinage de zéro de la fonction $g \circ f$.

- En déduire la limite: $\lim_{x \rightarrow 0} \frac{g \circ f(x) - 1 + \frac{x^2}{2}}{x^4}.$

Exercice 10:

Calculer les limites suivantes :

$$\lim_{x \rightarrow 0} \left(\frac{1}{\ln(x+1)} - \frac{1}{x} \right), \quad \lim_{x \rightarrow 0} \frac{e^{3x} \sin 3x}{\operatorname{sh}(-2x)}, \quad \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x} \right)^x.$$

Exo 1:

1) Montre que: $\text{Arc sin } x < \frac{x}{\sqrt{1-x^2}}$, $\forall x \in]0, 1[$:

On pose $f(x) = \text{Arc sin } x - \frac{x}{\sqrt{1-x^2}}$, $x \in]0, 1[$

f est définie et dérivable sur $]0, 1[$. On a:

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{\sqrt{1-x^2} - \frac{2x^2}{2\sqrt{1-x^2}}}{1-x^2} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}(1-x^2)}$$

$$= \frac{-x^2}{(\sqrt{1-x^2})(1-x^2)} < 0 \quad \left(\begin{array}{l} \text{car } x^2 > 0, \sqrt{1-x^2} > 0 \text{ et} \\ 1-x^2 > 0 \forall x \in]0, 1[\end{array} \right)$$

donc $f'(x) < 0 \Rightarrow f$ est décroissante
 $\forall x \in]0, 1[$.

ona: $x > 0 \Rightarrow f(x) < f(0) = 0$.

$$\Rightarrow \text{Arc sin } x - \frac{x}{\sqrt{1-x^2}} < 0$$

$$\Rightarrow \text{Arc sin } x < \frac{x}{\sqrt{1-x^2}}, \forall x \in]0, 1[$$

On suppose: $f(x) = \text{Arc tan } x - \frac{x}{1+x^2}$, $\forall x > 0$

f dérivable:

$$f'(x) = \frac{1}{1+x^2} - \frac{1+x^2 - 2x^2}{(1+x^2)^2} = \frac{1}{1+x^2} - \frac{1-x^2}{(1+x^2)^2}$$

$$= \frac{2x^2}{(1+x^2)^2} > 0$$

$\Rightarrow f$ est croissante

ona: $x > 0 \Rightarrow f(x) > f(0) = 0$

(1)

$$\Rightarrow \operatorname{Arctan} x - \frac{x}{1+x^2} > 0$$

$$\Rightarrow \operatorname{Arctan} x > \frac{x}{1+x^2}, \quad \forall x > 0.$$

* On suppose $f(x) = \operatorname{Arcsin} x + \operatorname{Arccos} x$, $x \in]-1, 1[$

f définie et dérivable $\Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$

$f'(x) = 0 \Rightarrow f(x)$ est constant $\forall x \in]-1, 1[$.

$$f(0) = \operatorname{Arcsin}(0) + \operatorname{Arccos}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

Alors : $\forall x \in]-1, 1[: f(x) = \frac{\pi}{2}$.

$$\Rightarrow \operatorname{Arcsin} x + \operatorname{Arccos} x = \frac{\pi}{2}, \quad \forall x \in]-1, 1[.$$

* On suppose $f(x) = \operatorname{Arctan} x + \operatorname{Arctan} \frac{1}{x}$.

$$f'(x) = \frac{1}{1+x^2} + \frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{1}{1+x^2} - \frac{x^2}{x^2(1+x^2)} = 0$$

$\Rightarrow f'(x) = 0 \Rightarrow f(x)$ est constante $\forall x > 0$.

$$f(1) = \operatorname{Arctan}(1) + \operatorname{Arctan}(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Alors : $\forall x > 0, f(x) = \frac{\pi}{2} \Rightarrow \operatorname{Arctan} x + \operatorname{Arctan} \frac{1}{x} = \frac{\pi}{2}$.

(2)

$$\text{On a : } n = \ln \left(\tan \left(\frac{y}{2} + \frac{\pi}{4} \right) \right)$$

$$* \cosh n = \frac{1}{2} (e^n + e^{-n}) = \frac{1}{2} \left(e^{\ln \left(\tan \left(\frac{y}{2} + \frac{\pi}{4} \right) \right)} + \frac{1}{e^{\ln \left(\tan \left(\frac{y}{2} + \frac{\pi}{4} \right) \right)}} \right)$$

$$= \frac{1}{2} \left[\tan \left(\frac{y}{2} + \frac{\pi}{4} \right) + \frac{1}{\tan \left(\frac{y}{2} + \frac{\pi}{4} \right)} \right]$$

$$= \frac{1}{2} \left[\frac{\sin \left(\frac{y}{2} + \frac{\pi}{4} \right)}{\cos \left(\frac{y}{2} + \frac{\pi}{4} \right)} + \frac{\cos \left(\frac{y}{2} + \frac{\pi}{4} \right)}{\sin \left(\frac{y}{2} + \frac{\pi}{4} \right)} \right]$$

$$= \frac{1}{2} \left(\frac{1}{\cos \left(\frac{y}{2} + \frac{\pi}{4} \right) \sin \left(\frac{y}{2} + \frac{\pi}{4} \right)} \right)$$

$$(2 \cos a \sin a = \sin 2a)$$

$$= \frac{1}{\sin \left(2 \left(\frac{y}{2} + \frac{\pi}{4} \right) \right)} = \frac{1}{\sin \left(y + \frac{\pi}{2} \right)}$$

$$(\sin \left(y + \frac{\pi}{2} \right) = \cos y)$$

$$= \frac{1}{\cos y}$$

$$* \sinh n = \frac{1}{2} (e^n - e^{-n}) = \frac{1}{2} \left(e^n - \frac{1}{e^n} \right)$$

$$= \frac{1}{2} \left[e^{\ln \left(\tan \left(\frac{y}{2} + \frac{\pi}{4} \right) \right)} - \frac{1}{e^{\ln \left(\tan \left(\frac{y}{2} + \frac{\pi}{4} \right) \right)}} \right]$$

(3)

$$= \frac{1}{2} \left[\tan \left(\frac{y}{2} + \frac{\pi}{4} \right) + \frac{1}{\tan \left(\frac{y}{2} + \frac{\pi}{4} \right)} \right]$$

$$= \frac{1}{2} \left[\frac{\sin \left(\frac{y}{2} + \frac{\pi}{4} \right)}{\cos \left(\frac{y}{2} + \frac{\pi}{4} \right)} - \frac{\cos \left(\frac{y}{2} + \frac{\pi}{4} \right)}{\sin \left(\frac{y}{2} + \frac{\pi}{4} \right)} \right]$$

$$= \frac{1}{2} \left[\frac{\sin^2 \left(\frac{y}{2} + \frac{\pi}{4} \right) - \cos^2 \left(\frac{y}{2} + \frac{\pi}{4} \right)}{\cos \left(\frac{y}{2} + \frac{\pi}{4} \right) \sin \left(\frac{y}{2} + \frac{\pi}{4} \right)} \right]$$

$$= \frac{-\cos 2 \left(\frac{y}{2} + \frac{\pi}{4} \right)}{2 \cos \left(\frac{y}{2} + \frac{\pi}{4} \right) \sin \left(\frac{y}{2} + \frac{\pi}{4} \right)}$$

$$(\sin^2 a - \cos^2 a = -\cos 2a)$$

$$(2 \cos a \sin a = \sin 2a)$$

$$= \frac{-\cos \left(y + \frac{\pi}{2} \right)}{\sin \left(y + \frac{\pi}{2} \right)} = \frac{\sin y}{\cos y} = \tan y$$

$$\begin{cases} -\cos \left(a + \frac{\pi}{2} \right) = \sin a \\ \sin \left(a + \frac{\pi}{2} \right) = \cos a \end{cases}$$

$$\star \quad \tanh x = \frac{\sinh x}{\cosh x} = \tan y \cdot \cos y$$

$$= \frac{\sin y}{\cancel{\cos y}} \cdot \cancel{\cos y} = \sin y$$

$$2/ \quad \lim_{n \rightarrow +\infty} e^{-n} (\cosh^3 n - \sinh^3 n) = \lim_{n \rightarrow +\infty} e^{-n} \frac{1}{8} [(e^n + e^{-n})^3 - (e^n - e^{-n})^3]$$

$$= \lim_{n \rightarrow +\infty} \frac{e^{-n}}{8} [(e^{3n} + 3e^n + 3e^{-n} + e^{-3n}) - (e^{3n} - 3e^n + 3e^{-n} - e^{-3n})]$$

$$= \lim_{n \rightarrow +\infty} \frac{e^{-n}}{8} (6e^n + 2e^{-3n}) = \lim_{n \rightarrow +\infty} \left(\frac{6}{8} + \frac{2e^{-4n}}{8} \right) = \frac{6}{8} = \boxed{\frac{3}{4}}$$

(4)

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} (n - \ln(e^n)) &= \lim_{n \rightarrow +\infty} (n - \ln(\frac{e^n + e^{-n}}{2})) \\
 &= \lim_{n \rightarrow +\infty} (n - \ln \frac{e^n}{2} (1 + e^{-2n})) \\
 &= \lim_{n \rightarrow +\infty} (\cancel{n} - \cancel{n} + \ln 2 + \ln(1 + e^{-2n})) \\
 &= \boxed{\ln 2}.
 \end{aligned}$$

Exo 3

La formule de Maclaurin est :

$$f(n) = f(0) + \frac{f'(0)}{1!}n + \frac{f''(0)}{2!}n^2 + \dots + \frac{f^{(n)}(0)}{n!}n^n + \frac{f^{(n+1)}(\theta n)}{(n+1)!}n^{n+1}$$

$0 < \theta < 1$.

Montrons que : $\forall n > 0, \quad n - \frac{n^2}{2} < \ln(1+n) < n$.

* $n - \frac{n^2}{2} < \ln(1+n) :$

$$f(n) = f(0) + \frac{f'(0)}{1!}n + \frac{f''(0)}{2!}n^2 + \frac{f^{(3)}(\theta n)}{3!}n^3, \quad 0 < \theta < 1.$$

$$f(n) = \ln(1+n) \Rightarrow f(0) = 0.$$

$$f'(n) = \frac{1}{1+n} \Rightarrow f'(0) = 1.$$

$$f''(n) = \frac{-1}{(1+n)^2} \Rightarrow f''(0) = -1.$$

$$f^{(3)}(n) = \frac{2}{(1+n)^3} \Rightarrow f^{(3)}(\theta n) = \frac{2}{(1+\theta n)^3}.$$

(5)

Apres: $f(n) = \ln(n+1) = n - \frac{1}{2}n^2 + \frac{1}{3!} \frac{2}{(1+\theta n)^3} n^3$

On a: $\frac{1}{3(1+\theta n)^3} > 0$ (car $n > 0$ et $0 < \theta < 1$)

$$\Rightarrow n - \frac{1}{2}n^2 + \frac{1}{3(1+\theta n)^3} n^3 > n - \frac{n^2}{2}$$

$$\Rightarrow \ln(1+n) > n - \frac{n^2}{2}$$

* $\ln(1+n) < n$:

$$f(n) = \ln(1+n) = f(0) + \frac{f'(0)}{1!} n + \frac{f''(\theta n)}{2!} n^2, \quad 0 < \theta < 1.$$

$$f(0) = 0, \quad f'(0) = 1, \quad f''(n) = \frac{-1}{(1+n)^2} \Rightarrow f''(\theta n) = \frac{-1}{(1+\theta n)^2}$$

$$f(n) = \ln(n+1) = n - \frac{1}{2(1+\theta n)^2} n^2, \quad 0 < \theta < 1.$$

on a: $\frac{-1}{2(1+\theta n)^2} n^2 < 0$ (car $n > 0$, et $0 < \theta < 1$)

$$\Rightarrow n - \frac{1}{2(1+\theta n)^2} n^2 < n \Rightarrow \ln(1+n) < n.$$

Apres: $n - \frac{n^2}{2} < \ln(1+n) < n. \quad \forall n > 0.$

calculons la limite,

$$\lim_{n \rightarrow 0} \frac{\ln(n+1) - n + \frac{n^2}{4}}{(\sin n)^2} = \lim_{n \rightarrow 0}$$

$$\frac{x - \frac{x^2}{2} + x^2 \varepsilon(n) - x + \frac{x^2}{4}}{(\sin x)^2}$$

$$= \lim_{n \rightarrow 0} \frac{-\frac{1}{4}x^2 + x^2 \varepsilon(n)}{(\sin x)^2} = \lim_{n \rightarrow 0} \frac{x^2 \left(-\frac{1}{4} + \varepsilon(n)\right)}{(\sin x)^2}$$

$$= \lim_{n \rightarrow 0} \frac{-\frac{1}{4} + \cancel{\varepsilon(n)}}{\cancel{\left(\frac{\sin n}{n}\right)^2}} = \boxed{-\frac{1}{4}}$$

Exo 5 :

$$1) f(n) = e^n = f(0) + \frac{f'(0)}{1!}n + \frac{f''(0)}{2!}n^2 + \dots + \frac{f^{(n)}(0)}{n!}n^n + \frac{f^{(n+1)}(\theta n)}{(n+1)!}n^{n+1}, \quad 0 < \theta < 1.$$

$$= \sum_{k=0}^n \frac{f^{(k)}(0)}{k!}n^k + \frac{f^{(n+1)}(\theta n)}{(n+1)!}n^{n+1}, \quad 0 < \theta < 1.$$

$$\text{On a : } f^{(n)}(n) = e^n.$$

$$\Rightarrow \begin{cases} f^{(n)}(0) = 1 \\ \text{et} \\ f^{(n+1)}(\theta n) = e^{\theta n} \end{cases}$$

$$\text{Alors : } f(n) = e^n = 1 + \frac{n}{1!} + \frac{n^2}{2!} + \dots + \frac{n^n}{n!} + \frac{e^{\theta n}}{(n+1)!}n^{n+1}, \quad 0 < \theta < 1.$$

$$= \sum_{k=0}^n \frac{n^k}{k!} + \frac{e^{\theta n}}{(n+1)!}n^{n+1}, \quad 0 < \theta < 1.$$

(7)

*) Montre que :

$$1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{e}{(n+1)!}$$

On a : pour $n=1$ dans la formule précédente :

$$f(1) = e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{e^\theta}{(n+1)!}, \quad 0 < \theta < 1.$$

$$= \sum_{k=0}^n \frac{1}{k!} + \frac{e^\theta}{(n+1)!}, \quad 0 < \theta < 1.$$

on a : $0 < \theta < 1 \Rightarrow 1 < e^\theta < e \Rightarrow \frac{1}{(n+1)!} < \frac{e^\theta}{(n+1)!} < \frac{e}{(n+1)!}$

$$\Rightarrow \frac{1}{(n+1)!} < e - \sum_{k=0}^n \frac{1}{k!} < \frac{e}{(n+1)!}$$

$$\Rightarrow 0 < e - \sum_{k=0}^n \frac{1}{k!} < \frac{e}{(n+1)!}$$

$$\Rightarrow \sum_{k=0}^n \frac{1}{k!} < e < \frac{e}{(n+1)!} + \sum_{k=0}^n \frac{1}{k!}$$

$$e^n = \sum_{k=0}^n \frac{1}{k!} + \frac{e^\theta}{(n+1)!}$$

$$\Rightarrow e^n - \sum_{k=0}^n \frac{1}{k!} = \frac{e^\theta}{(n+1)!}$$

donc : $1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} < e < 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{e}{(n+1)!}$

*) Déduire la limite de la suite $u_n = \sum_{k=0}^n \frac{1}{k!}$

On a : $0 < e - \sum_{k=0}^n \frac{1}{k!} < \frac{e}{(n+1)!}$

passage à la limite :

$$0 < \lim_{n \rightarrow +\infty} \left(e - \sum_{k=0}^n \frac{1}{k!} \right) < \lim_{n \rightarrow +\infty} \frac{e}{(n+1)!} = 0$$

$$\Rightarrow \lim_{n \rightarrow +\infty} \left(e - \sum_{k=0}^n \frac{1}{k!} \right) = 0 \Rightarrow \sum_{k=0}^n \frac{1}{k!} = e$$

(8)

$$\Rightarrow \lim_{n \rightarrow +\infty} u_n = e$$

6:

$$\text{On a: } f(x) = P_n(x+1) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta_n)}{(n+1)!}x^{n+1}, \quad 0 < \theta < 1.$$

et on a: $f^{(n)}(x) = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n}$

$$\Rightarrow f(x) = P_n(x+1) = x - \frac{1}{2}x^2 + \dots + \frac{(-1)^{n-1}}{n!}x^n + \frac{(-1)^n}{(n+1)(1+\theta)^{n+1}}x^{n+1}$$

$\underbrace{\frac{f^{(n+1)}(\theta_n)}{(n+1)!}x^{n+1}}_{\frac{f^{(n+1)}(\theta_n)}{(n+1)!}x^{n+1}}$

pour $x=1$:

$$f(1) = P_n(2) = 1 - \frac{1}{2} + \dots + \frac{(-1)^{n-1}}{n!} + \frac{(-1)^n}{(n+1)} \frac{1}{(1+\theta)^{n+1}}$$

$$\Rightarrow 1 - \frac{1}{2} + \dots + \frac{(-1)^{n-1}}{n!} = P_n(2) - \frac{(-1)^n}{(n+1)} \frac{1}{(1+\theta)^{n+1}}$$

$$\Rightarrow \lim_{n \rightarrow +\infty} \left(1 - \frac{1}{2} + \dots + \frac{(-1)^{n-1}}{n!} \right) = \lim_{n \rightarrow +\infty} \left(P_n(2) - \frac{(-1)^n}{(n+1)} \frac{1}{(1+\theta)^{n+1}} \right)$$

$= \ln(2).$

Exo 6:

$$U_n = \prod_{k=1}^n \left(1 + \frac{k}{n^2} \right), \quad \forall n \in \mathbb{N}^*$$

On pose $V_n = \ln U_n \Rightarrow V_n = \ln \prod_{k=1}^n \left(1 + \frac{k}{n^2} \right)$

$$\Rightarrow V_n = \sum_{k=1}^n \ln \left(1 + \frac{k}{n^2} \right).$$

D'après Exo 3 :

$$\frac{k}{n^2} - \frac{k^2}{2n^4} < \ln\left(1 + \frac{k}{n^2}\right) < \frac{k}{n^2}$$

$$\Rightarrow \sum_{k=1}^n \frac{k}{n^2} - \frac{k^2}{2n^4} < \sum_{k=1}^n \ln\left(1 + \frac{k}{n^2}\right) < \sum_{k=1}^n \frac{k}{n^2}$$

$$\Rightarrow \cancel{\frac{1}{n^2} \sum_{k=1}^n k} - \cancel{\frac{1}{2n^4} \sum_{k=1}^n k^2} < \sum_{k=1}^n \ln\left(1 + \frac{k}{n^2}\right) < \cancel{\frac{1}{n^2} \sum_{k=1}^n k}$$

$$\Rightarrow \frac{1}{n^2} \sum_{k=1}^n k - \frac{1}{2n^4} \sum_{k=1}^n k^2 < \sum_{k=1}^n \ln\left(1 + \frac{k}{n^2}\right) < \frac{1}{n^2} \sum_{k=1}^n k$$

$$\text{On a : } \sum_{k=1}^n k = \frac{n(n+1)}{2}$$

$$\text{et on a : } k^2 \leq n^2 \Rightarrow \sum_{k=1}^n k^2 \leq \sum_{k=1}^n n^2 = n \cdot n^2$$

$$\Rightarrow \frac{1}{2n^4} \sum_{k=1}^n k^2 \leq \frac{1}{2n^4} n \cdot n^2 = \frac{1}{2n}$$

$$\Rightarrow \frac{-1}{2n^4} \sum_{k=1}^n k^2 \geq -\frac{1}{2n}$$

$$\text{Alors : } \frac{1}{n^2} \frac{n(n+1)}{2} - \frac{1}{2n} < \sum_{k=1}^n \ln\left(1 + \frac{k}{n^2}\right) < \frac{1}{n^2} \frac{n(n+1)}{2}$$

passage à la limite,

$$\lim_{n \rightarrow +\infty} \left(\cancel{\frac{1}{n^2} \frac{n(n+1)}{2}} - \cancel{\frac{1}{2n}} \right) < \lim_{n \rightarrow +\infty} \sum_{k=1}^n \ln\left(1 + \frac{k}{n^2}\right) < \lim_{n \rightarrow +\infty} \cancel{\frac{1}{n^2} \frac{n(n+1)}{2}}$$

$$\Rightarrow \lim_{n \rightarrow +\infty} \sum_{k=1}^n \ln\left(1 + \frac{k}{n^2}\right) = \frac{1}{2} \Rightarrow \lim_{n \rightarrow +\infty} V_n = \frac{1}{2}$$

$$(10) \Rightarrow \lim_{n \rightarrow +\infty} V_n = \lim_{n \rightarrow +\infty} e^{V_n} = e^{\frac{1}{2}}$$

$$1) f(n) = n (\ln n)^{\frac{1}{n}}, \quad n_0 = 0, \quad n = 3.$$

$$= n e^{\frac{1}{n} \ln(\ln n)}$$

$$\text{On a: } \ln n = 1 + \frac{n^2}{2} + o(n^2) \Rightarrow \ln(\ln n) = \ln\left(1 + \frac{n^2}{2} + o(n^2)\right)$$

$$\text{On pose: } y = \frac{n^2}{2} + o(n^2). \quad (n \rightarrow 0 \Rightarrow y \rightarrow 0)$$

$$\text{Donc: } \ln(1+y) = y - \frac{1}{2}y^2 + o(y^2).$$

$$\Rightarrow \ln\left(1 + \frac{n^2}{2} + o(n^2)\right) = \ln(\ln n) = \frac{n^2}{2} - \frac{1}{8}n^4 + o(n^4).$$

$$\Rightarrow \frac{1}{n} \ln(\ln n) = \frac{n}{2} - \frac{1}{8}n^3 + o(n^3).$$

$$e^{\frac{1}{n} \ln(\ln n)} = e^{\frac{n}{2} - \frac{1}{8}n^3 + o(n^3)}$$

$$\text{On pose } z = \frac{n}{2} - \frac{1}{8}n^3 + o(n^3) \quad (n \rightarrow 0 \Rightarrow z \rightarrow 0).$$

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{3!}z^3 + o(z^3).$$

$$\Rightarrow e^{\frac{1}{n} \ln(\ln n)} = 1 + \frac{n}{2} - \frac{1}{8}n^3 + \frac{1}{2} \left(\frac{n}{2} - \frac{1}{8}n^3 \right)^2 + \frac{1}{6} \left(\frac{n}{2} - \frac{1}{8}n^3 \right)^3 + o(n^3)$$

$$= 1 + \frac{1}{2}n + \frac{1}{8}n^2 + \frac{5}{48}n^3 + o(n^3)$$

$$\Rightarrow f(n) = n e^{\frac{1}{n} \ln(\ln n)} = n \left(1 + \frac{1}{2}n + \frac{1}{8}n^2 + \frac{5}{48}n^3 + o(n^3) \right)$$

$$= n + \frac{1}{2}n^2 + \frac{1}{8}n^3 - \frac{5}{48}n^3 + o(n^3).$$

$$2) f(n) = \ln(1 + \sin x), \quad x_0 = 0, \quad n = 3.$$

$$\text{On a : } \sin x = x - \frac{1}{3!} x^3 + O(x^3).$$

$$\Rightarrow f(n) = \ln\left(1 + x - \frac{1}{3!} x^3 + O(x^3)\right), \text{ on pose :}$$

$$y = x - \frac{1}{3!} x^3 + O(x^3)$$

$$f(n) = \ln(1+y) = y - \frac{1}{2} y^2 + \frac{1}{3} y^3 + O(y^3).$$

$$\Rightarrow f(n) = \ln(1 + \sin x) = x - \frac{1}{6} x^3 - \frac{1}{2} \left(x - \frac{1}{6} x^3\right)^2 + \frac{1}{3} \left(x - \frac{1}{6} x^3\right)^3 + O(x^3)$$

$$= x - \frac{1}{6} x^3 - \frac{1}{2} x^2 + \frac{1}{3} x^3 + O(x^3)$$

$$= x - \frac{1}{2} x^2 + \frac{1}{6} x^3 + O(x^3).$$

$$3) f(n) = \frac{1+x+x^2}{1-x+x^2}, \quad x_0 = 0, \quad n = 4:$$

$$\begin{array}{r|l} 1+x+x^2 & 1-x+x^2 \\ \hline 1-x+x^2 & 1+2x+2x^2-2x^4 \\ \hline 2x & \\ 2x-2x^2+2x^3 & \\ \hline 2x^2-2x^3 & \\ 2x^2-2x^3+2x^4 & \\ \hline -2x^4 & \end{array}$$

$$\text{donc : } f(n) = \frac{1+x+x^2}{1-x+x^2} = 1 + 2x + 2x^2 - 2x^4 + O(x^4)$$

$$f(x) = \cos(\sin x) \quad , \quad x_0 = 0, \quad n = 4$$

$$\text{on a : } \sin x = x - \frac{1}{6}x^3 + O(x^5).$$

$$\text{on pose : } y = x - \frac{1}{6}x^3.$$

$$\Rightarrow \cos y = 1 - \frac{1}{2}y^2 + \frac{1}{24}y^4 + O(y^6).$$

$$\cos(\sin x) = 1 - \frac{1}{2}\left(x - \frac{1}{6}x^3\right)^2 + \frac{1}{24}\left(x - \frac{1}{6}x^3\right)^4 + O(x^6)$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{6}x^4 + \frac{1}{24}x^4 + O(x^6)$$

$$\Rightarrow f(x) = \cos(\sin x) = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + O(x^6).$$

$$5) f(x) = \frac{x}{1-x^2} \quad , \quad x_0 = 0, \quad n = 4.$$

$$= x \cdot \frac{1}{1-x^2} = x(1+x^2+O(x^4)) = x+x^3+O(x^5).$$

$$6) f(x) = \ln\left(\frac{\sin x}{x}\right) \quad , \quad x_0 = 0, \quad n = 4$$

$$\text{on a : } \sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7) \Rightarrow$$

$$\frac{\sin x}{x} = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + O(x^6)$$

$$\Rightarrow f(x) = \ln\left(\frac{\sin x}{x}\right) = \ln\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + O(x^6)\right)$$

$$\ln(1+y) = y - \frac{1}{2}y^2 + O(y^3).$$

$$\Rightarrow f(x) = \ln\left(\frac{\sin x}{x}\right) = -\frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{2}\left(-\frac{1}{6}x^2 + \frac{1}{120}x^4\right)^2 + O(x^6)$$

$$\Rightarrow f(x) = -\frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{72}x^4 + O(x^6) = -\frac{1}{6}x^2 - \frac{1}{180}x^4 + O(x^6)$$

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$$7/ \quad f(n) = \frac{1+2n+n^3}{n^3+n^5}, \quad n_0 = 0, \quad n = 2$$

$$= \frac{1}{n^3} \left(\frac{1+2n+n^3}{1+n^2} \right)$$

$$= \frac{1}{n^3} (1+2n+n^3) \left(\frac{1}{1+n^2} \right).$$

$$\Rightarrow f(n) = \frac{1}{n^3} (1+2n+n^3) (1-n^2+n^4+O(n^4))$$

$$= \frac{1}{n^3} (1 - n^2 + n^4 + 2n - 2n^3 + 2n^5 + n^3 - n^5 + O(n^5))$$

$$= \frac{1}{n^3} + \frac{2}{n^2} - \frac{1}{n} - 1 + n + n^2 + O(n^2).$$

$$8/ \quad f(n) = \exp(\sqrt{n}), \quad n_0 = 1, \quad n = 3.$$

suppose: $y = n-1 \Rightarrow n = y+1$ (si $n=1 \Rightarrow y=0$)

Donc: $e^{\sqrt{n}} = e^{\sqrt{y+1}}$

On a: $\sqrt{y+1} = 1 + \frac{y}{2} - \frac{1}{8}y^2 + \frac{1}{16}y^3 + O(y^3)$

$$e^{\sqrt{y+1}} = e^{1 + \frac{y}{2} - \frac{1}{8}y^2 + \frac{1}{16}y^3 + O(y^3)} = e \cdot e^{\frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 + O(y^3)}.$$

suppose: $z = \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 + O(y^3)$. (si $y \rightarrow 0 \Rightarrow z \rightarrow 0$).

$$\Rightarrow e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + O(z^3)$$

$$\Rightarrow e^{\sqrt{y+1}} = e \cdot e^z = e \left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + O(z^3) \right).$$

$$= e \left(1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 + \frac{1}{2} \left(1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 \right)^2 + \frac{1}{6} \left(1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 \right)^3 + O(y^3) \right).$$

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$$e \left[1 + \frac{1}{2}y - \frac{1}{8}y^2 + \frac{1}{16}y^3 + \frac{1}{8}y^2 - \frac{1}{32}y^3 - \frac{1}{32}y^3 + \frac{1}{48}y^3 + o(y^3) \right].$$

$$= e \left[1 + \frac{1}{2}y + \frac{1}{48}y^3 + o(y^3) \right].$$

$$\Rightarrow f(x) = e^{\sqrt{x}} = e \left[1 + \frac{1}{2}(x-1) + \frac{1}{48}(x-1)^3 + o((x-1)^3) \right].$$

$$= e + \frac{e}{2}(x-1) + \frac{e}{48}(x-1)^3 + o((x-1)^3).$$

g) $f(x) = \frac{x^3+2}{x-1}$, $x_0 = +\infty$, $n=3$

suppose: $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$ (si $x \rightarrow \infty, \Rightarrow y \rightarrow 0$).

$$f(x) = \frac{x^3+2}{x-1} = \frac{\left(\frac{1}{y}\right)^3 + 2}{\left(\frac{1}{y} - 1\right)} = \frac{\frac{1+2y^3}{y^3}}{\frac{1-y}{y}} = \frac{1}{y^2} \left(\frac{1+2y^3}{1-y} \right).$$

$$= \frac{1}{y^2} (1+2y^3) \left(\frac{1}{1-y} \right).$$

$$= \frac{1}{y^2} (1+2y^3) (1+y+y^2+y^3+y^4+y^5+o(y^5)).$$

$$= \frac{1}{y^2} (1+y+y^2+y^3+y^4+y^5+2y^3+2y^4+2y^5+o(y^5)).$$

$$= \frac{1}{y^2} (1+y+y^2+3y^3+3y^4+3y^5+o(y^5)).$$

$$= \frac{1}{y^2} + \frac{1}{y} + 1 + 3y + 3y^2 + 3y^3 + o(y^3).$$

$$= 1 + x^2 + x + \frac{3}{x} + \frac{3}{x^2} + \frac{3}{x^3} + o\left(\left(\frac{1}{x}\right)^3\right).$$

Exo 9:

$$* (g \circ f)(n) = g(f(n)) = 1 - \frac{1}{2} \left(n - \frac{n^3}{6} \right)^2 + \frac{1}{24} \left(n - \frac{n^3}{6} \right) + o(n^4).$$

$$= 1 + \frac{1}{2} n^2 + \frac{1}{6} n^4 + \frac{1}{24} n^4 + o(n^4).$$

$$= 1 - \frac{1}{2} n^2 + \frac{5}{24} n^4 + o(n^4).$$

$$\lim_{n \rightarrow 0} \frac{g \circ f(n) - 1 + \frac{n^2}{2}}{n^4} = \lim_{n \rightarrow 0} \frac{\frac{5}{24} n^4 + o(n^4)}{n^4}$$

$$= \lim_{n \rightarrow 0} \frac{\cancel{n^4} \left(\frac{5}{24} + \frac{o(n^4)}{\cancel{n^4}} \right)}{\cancel{n^4}} = \frac{5}{24}.$$

Exo 10:

$$1) \lim_{n \rightarrow 0} \left(\frac{1}{\ln(n+1)} - \frac{1}{n} \right) = \lim_{n \rightarrow 0} \left(\frac{1}{n - \frac{n^2}{2} + o(n^2)} - \frac{1}{n} \right),$$

$$= \lim_{n \rightarrow 0} \frac{1}{n} \left(\frac{1}{1 - \frac{n}{2}} - 1 \right) = \lim_{n \rightarrow 0} \frac{1}{n} \left(\frac{1}{\left(\frac{1}{2-n} + o(n) \right)} - 1 \right)$$
$$= \lim_{n \rightarrow 0} \left(\frac{1}{2} + \frac{o(n)}{n} \right) = \frac{1}{2}$$

$$2) \lim_{n \rightarrow 0} \frac{e^{3n} \sin 3n}{\operatorname{sh}(-2n)} = \lim_{n \rightarrow 0} \frac{(1 + 3n + o(n)) (3n + o(n))}{-2n + o(n)}$$
$$= \lim_{n \rightarrow 0} \frac{3n + o(n)}{-2n + o(n)} = \lim_{n \rightarrow 0} \left(-\frac{3}{2} + \frac{o(n)}{n} \right) = -\frac{3}{2}.$$

$$\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n \quad (\text{suppose: } y = \frac{1}{n})$$

$$= \lim_{y \rightarrow 0} (1+y)^{\frac{1}{y}} = \lim_{y \rightarrow 0} e^{\frac{1}{y} \ln(1+y)}$$

$$= \lim_{y \rightarrow 0} e^{\frac{1}{y} (y - \frac{1}{2}y^2 + o(y^2))}$$

$$= \lim_{y \rightarrow 0} e^{1 - \frac{1}{2}y + o(y)}$$

$$= \lim_{y \rightarrow 0} (e \cdot e^{-\frac{1}{2}y + o(y)})$$

$$= e.$$

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